Image Restoration Using the Damped Richardson-Lucy Method

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Abstract. A modification of the Richardson-Lucy iteration that reduces noise amplification in restored images is described.

1. Introduction

The Richardson-Lucy (R-L) algorithm (Richardson 1972, Lucy 1974) is the technique most widely used for restoring HST images. The standard R-L method has a number of characteristics that make it well-suited to HST data:

- The R-L iteration converges to the maximum likelihood solution for Poisson statistics in the data (Shepp and Vardi 1982), which is appropriate for optical data with noise from counting statistics.
- The R-L method forces the restored image to be non-negative and conserves flux both globally and locally at each iteration.
- The restored images are robust against small errors in the point-spread function (PSF).
- Typical R-L restorations require a manageable amount of computer time.

The R-L iteration can be derived very simply if we start with the imaging equation and the equation for Poisson statistics. The imaging equation tells how the true image is blurred by the PSF:

\[ I(i) = \sum_{j} P(\alpha | j) O(j) , \]

where \( O \) is the unblurred object, \( P(\alpha | j) \) is the PSF (the fraction of light coming from true location \( j \) that gets scattered into observed pixel \( i \)), and \( I \) is the noiseless blurry image. If the PSF does not vary with position in the image, then \( P(\alpha | j) = P(i - j) \) and the sum becomes a convolution. The probability of getting \( N \) counts in a pixel when the mean expected number of counts is \( \bar{N} \) is given by the Poisson distribution:

\[ P(N | \bar{N}) = \frac{e^{-\bar{N}} \bar{N}^N}{N!} \]  

and so the joint likelihood \( \mathcal{L} \) of getting the observed counts \( D(i) \) in each pixel given the expected counts \( I(i) \) is

\[ \ln \mathcal{L} = \sum_{i} D(i) \ln I(i) - I(i) - \ln D(i)! . \]  

The maximum likelihood solution occurs where all partial derivatives of \( \mathcal{L} \) with respect to \( O(j) \) are zero:

\[ \frac{\partial \ln \mathcal{L}}{\partial O(j)} = 0 = \sum_{i} \left[ \frac{D(i)}{I(i)} - 1 \right] P(i | j) . \]
The R-L iteration is simply

\[ O_{\text{new}}(j) = O(j) + \frac{\sum_i P(i|j) D(i)}{\sum_i P(i|j)} I(i). \]  

(5)

It is clear from a comparison of Eqs. (4) and (5) that if the R-L iteration converges (as has been proven by Shepp & Vardi 1982), meaning that the correction factor approaches unity as the iterations proceed, then it must indeed converge to the maximum likelihood solution for Poisson statistics in the data.

## 2. Noise Amplification

Despite its advantages, the R-L method has some serious shortcomings. In particular, noise amplification can be a problem. This is a generic problem for all maximum likelihood techniques, which attempt to fit the data as closely as possible. If one performs many R-L iterations on an image containing an extended object such as a galaxy, the extended emission usually develops a “speckled” appearance (Fig. 1). The speckles are not representative of any real structure in the image, but are instead the result of fitting the noise in the data too closely. In order to reproduce a small noise bump in the data it is necessary for the unblurred image to have a very large noise spike; pixels near the bright spike must then be very black (near zero brightness) in order to conserve flux.

The only limit on the amount of noise amplification in the R-L method is the requirement that the image not become negative. Thus, once the compensating holes in the image are pushed down to zero flux, nearby spikes cannot grow any further and noise amplification ceases. The positivity constraint alone is sufficient to control noise amplification in images of star fields on a black background; in that case one can perform thousands of R-L iterations without generating an unacceptable amount of noise. However, for smooth objects observed at low signal-to-noise, even a modest number of R-L iterations (20–30) can produce objectionable noise.

The usual practical approach to limiting noise amplification is simply to stop the iteration when the restored image appears to become too noisy. However, the question of where to stop is a difficult one. The approach suggested by Lucy (1974) was to stop when the reduced \( \chi^2 \) between the data and the blurred model is about 1 per degree of freedom. Unfortunately, one does not really know how many degrees of freedom have been used to fit the data. If one stops after a very few iterations then the model is still very smooth and the resulting \( \chi^2 \) should be comparable to the number of pixels. If one performs many iterations, however, then the model image develops a great deal of structure and so the effective number of degrees of freedom used is large; in that case, the fit to the data ought to be considerably better. There is no criterion for the R-L method that tells how close the fit ought to be. Note that there is such a criterion built in to the MEMSYS 5 maximum entropy package (Gull and Skilling 1991), and the pixon approach of Piña and Puetter (1993) uses similar ideas.

Another problem is that the answer to the question of how many iterations to perform often is different for different parts of the image. It may require hundreds of iterations to get a good fit to the high signal-to-noise image of a bright star, while a smooth, extended object may be fitted well after only a few iterations. In Fig. 1, note how the images of both the central star and the bright star at the top center continue to improve as the number of iterations increases, while the noise amplification in the extended nebulosity is getting much worse. Thus, one would like to be able to slow or stop the iteration automatically in regions where a smooth model fits the data adequately, while continuing to iterate in regions where there are sharp features (edges or point sources).

Another approach to controlling noise amplification is to smooth the final restored image. This method has been developed and mathematically justified by Snyder and his co-workers (Snyder and Miller 1985, Snyder et al. 1987). Unfortunately, for HST images the amount of smoothing required to reduce the noise amplification is very large. Fig. 2 shows the effect of various amounts
Figure 1.   R-L restoration of simulated $85 \times 85$ pixel HST PC observation of a planetary nebula. Restored and true images are $256 \times 256$ (method of White 1990 was used to restore image with finer pixels than data.) As the number of iterations (shown at top of each image) increases, the images of bright stars improve, but noise is amplified unacceptably in the nebulosity.

Figure 2.   Results of smoothing 400-iteration R-L restoration from Fig. 1 with a Gaussian. The width of the Gaussian is $\sigma = 1$, 2, and 3 pixels for the three cases shown. Images of stars are grossly blurred when the R-L image is smoothed heavily enough to substantially reduce the noise in the nebulosity.
of smoothing on the restored planetary nebula image. By the time the noise amplification in the nebulosity has been reduced to a visually acceptable level, the images of stars have been grossly blurred. For most purposes, one must pay far too high a price to avoid noise amplification using this method.

3. The Damped R-L Iteration

An effective approach to such an adaptive stopping criterion is to modify the likelihood function of Eq. (3) so that it becomes flatter in the vicinity of a good fit. The approach I have taken is to use a likelihood function that is identical to Eq. (3) when the difference between the blurred model \( I \) and the data \( D \) is large compared with the noise, but that is essentially constant when the difference is smaller than the noise.

There are two important advantages to using a modified form of the likelihood function to control noise amplification:

- The end point of the iteration is well-defined, allowing properties of the final solution to be studied.
- Acceleration techniques may be used to reach the solution more quickly without changing the final answer.

It is appropriate to prevent noise amplification by recognizing its effects in the data domain rather than by applying regularizing constraints on the restored image. Residuals that are smaller than one expects based on the known noise properties of the data should be avoided by using a likelihood function that does not reward overfitting the data. Of course, it may be necessary to use constraints on the model image in order to select one of the many possible restored images that fit the data adequately well, but in my opinion it is important to first establish better goodness-of-fit criteria so that the set of possible restored images is as small as possible before applying regularization to select one of those images. The work of Piña & Puettter (1992) on the Maximum Residual Likelihood criterion for data fitting is a notable attempt in this direction.

The damped R-L iteration starts from the likelihood function

\[
\ln \mathcal{L} = \sum_i f(U_i),
\]

where \( U_i = \frac{2}{T^2} \left[ D(i) \ln \frac{I(i)}{D(i)} - I(i) + D(i) \right] \)

and

\[
f(x) = \begin{cases} 
\frac{N}{N+1} \left( 1 - x^{N+1} \right) + x^N, & x < 1 \\
x, & x \geq 1
\end{cases}
\]

\( f(x) \) is the “damping function”. It is chosen to be a simple function that is linearly proportional to \( x \) for \( x > 1 \), is approximately constant for \( x \sim 0 \), and has continuous first and second derivatives at \( x = 1 \) (this allows acceleration techniques to be used to speed convergence of the method). The constant \( N \) determines how suddenly the function \( f \) becomes flat for \( x < 1 \). For \( N = 0 \), \( f(x) = x \) and there is no flattening at all. The larger the value of \( N \), the flatter is the function. The results reported in this paper use \( N = 10 \), but the value of \( N \) has little effect on the results as long as it is larger than a few.

\( U_i \) is a slightly modified version of the \( \ln \mathcal{L} \) function of Eq. (3). The constants and multiplicative factors are chosen so that the expected value of \( U \) in the presence of Poisson noise is unity if the threshold \( T = 1 \). The threshold \( T \) then determines at what level the damping turns on: if \( T = 1 \) the damping occurs at \( 1 \sigma \), if \( T = 2 \) at \( 2 \sigma \), etc.
Figure 3. Restoration of data from Fig. 1 using the damped iteration with noise thresholds of 2 and 3σ. As number of iterations increases, images of bright stars continue to improve, but noise amplification is much better controlled than for standard R-L iteration.

With this new likelihood function, we can follow the steps outlined above to derive the damped R-L iteration:

\[
O_{\text{new}}(j) = \frac{\sum P(i|j) \left[ 1 + \tilde{U}_i \frac{D(i) - I(i)}{I(i)} \right]}{\sum P(\tilde{d}|j)},
\]

where

\[
\tilde{U}_i = \min(U_{i,1}).
\]

Note that in regions where the data and model do not agree, \(\tilde{U} = 1\) and so this iteration is exactly the same as the standard R-L iteration. In regions where the data and model do agree, however, the second term gets multiplied by a factor which is less than 1, and the ratio of the numerator and denominator approaches unity. This has exactly the desired character: it damps changes in the model in regions where the differences are small compared with the noise.

Fig. 3 shows the results of applying the damped iteration to the planetary nebula data of Fig. 1. Note that the noise amplification in the nebulosity has been greatly reduced, but that the stars are still very sharp. Fig. 4 shows the result of applying both the standard R-L method and the damped iteration to HST observations of Saturn; again, the noise in the restored image is greatly reduced without significantly blurring the sharp edges of the planet and its rings. This is perhaps more easily seen in Fig. 5, which shows the brightness of the restored image along a cut through the planet and rings. Sharp features such as narrow gaps in the rings are essentially identical in the R-L and damped images, but the disk of the planet is much smoother in the damped image. Note that the
Figure 4. HST observations of Saturn and restorations using the R-L iteration ($T = 0$) and the damped iteration with noise thresholds of 1 and $2\sigma$. 100 accelerated iterations were used in each case.

Figure 5. Cut through the rings and disk from upper right to lower left for the restored Saturn images in Fig. 4.
mean brightness in the disk of the planet is the same in the damped and R-L images, indicating good photometric linearity.

4. Summary

This paper describes an iterative image restoration technique that closely resembles the R-L iteration but that avoids the amplification of noise that occurs in the R-L iteration. I call this the “damped” R-L iteration because the modification appears as a damping factor that slows changes in regions of the model image that fit the data well while allowing the model to continue to improve in regions where it fits the data less well.

The approach used is to modify the likelihood function so that it is “flatter” in the vicinity of a good fit. It is unlikely that this ad hoc approach represents the best solution to the noise amplification problem, but it does have the advantages that it is easy to implement and robust, and it appears to produce better results than the R-L method in many cases. The method produces restored images that have good photometric linearity and little bias.

References