Fisher Matrices for data with errors in both variables

A.F. Heavens

*Imperial Centre for Inference and Cosmology, Department of Physics, Imperial College, Blackett Laboratory, Prince Consort Road, London SW7 2AZ, U.K.*

M. Seikel

*The UCT Astrophysics, Cosmology and Gravity Centre, Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch 7701, Cape Town, South Africa*

B.D. Nord

*Department of Physics, University of Michigan, Ann Arbor, Michigan, United States*

M. Aich

*School of Mathematics, Statistics & Computer Science, University of KwaZulu-Natal, Durban 4000, South Africa*

Y. Bouffanais

*Imperial Centre for Inference and Cosmology, Department of Physics, Imperial College, Blackett Laboratory, Prince Consort Road, London SW7 2AZ, U.K.*

B.A. Bassett

*African Institute for Mathematical Sciences, 6 Melrose Road, Muizenberg, 7945, South Africa*

*Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch, Cape Town, 7700, South Africa*

*South African Astronomical Observatory, Observatory Road, Observatory, Cape Town, 7935, South Africa*

**Summary.** The Fisher Information Matrix formalism (Fisher, 1935) is extended to cases where both elements of each data pair have errors, with arbitrary correlations.
between points. The analysis applies if all errors are gaussian, and if the errors in
the independent variable are small, both in comparison with the scale over which the
expected signal changes, and with the width of the prior distribution. This generalises
the Fisher Matrix approach, which normally only considers errors in the ordinate. In
this work, we include errors from observable data elements through marginalising
over latent variables, effectively employing a Bayesian hierarchical model, and deriv-
ing the Fisher Matrix for this more general case. The methods here also extend to
likelihood surfaces which are not gaussian in the parameter space, and so techniques
such as DALI (Derivative Approximation for Likelihoods) can be generalised straight-
forwardly to include arbitrary gaussian data error covariances. For simple mock data
and theoretical models, we compare to Markov Chain Monte Carlo experiments, il-
lustrating the method with cosmological supernova data. We also include the new
method in the Fisher4Cast software.

**Keywords:** statistics: general — statistics: Fisher matrix — cosmology: fore-
casts

### 1. Introduction

The Fisher Information Matrix or simply Fisher Matrix has become one of the
most widely used statistical tools for forecasting the errors in parameter estimation
problems. It provides lower limits on the variances, and the expected covariances of
estimates of model parameters from maximum likelihood, or maximum posterior,
techniques, for a given experimental design. The formalism assumes gaussianity
in two respects: the data are assumed to be jointly gaussian-distributed, and the
posterior for the parameters is also assumed to be gaussian. For data pairs \{X_i, Y_i\}
with no errors in X, the problem was solved many years ago (Fisher 1935). The
main value of the Fisher matrix technique is in being able to obtain error forecasts
without any data, real or simulated, and is generally much faster than computing
full posterior distributions with simulations (Acquaviva et al. 2012; Bassett et al.
2009). It is however only a first step, as it assumes the posteriors are well described
by multivariate gaussian distributions, and this may not hold (e.g., Wolz et al.}
2012), when more sophisticated analysis may be required, but it is still a very valuable tool for experimental design. Furthermore, more sophisticated forecasts for likelihood surfaces which are non-gaussian in the parameter space now exist (Sellentin et al., 2014).

From the initial derivations of the Fisher Matrix in the cosmological context (Vogeley & Szalay, 1996; Tegmark, Taylor, & Heavens, 1997), we have arrived today at very mature applications and implementations (e.g., Bassett et al., 2009; Coe, 2009; Refregier et al., 2011). The Fisher Matrix has been useful in proposals and projections for surveys, such as for the Cosmic Microwave Background (Taylor et al., 1997), spectroscopic galaxy surveys (Schlegel et al., 2011), the Dark Energy Survey (DES Collaboration, 2005), large-scale structure (Cunha, 2009), and in the broader discussion of the investigation of Dark Energy (Albrecht et al., 2006) and estimation of neutrino masses with the future European Space Agency Euclid mission (Kitching et al., 2008).

For the purposes of review and later reference in this work, we summarise the basic Fisher Matrix formalism. We begin with the likelihood of a set of data, \( d \) given (or conditional upon) a set of model parameters, represented by a vector \( \theta \):

\[
p(d|\theta).
\]

In the traditional approach, \( d \) represents only the ordinates, \( Y \). Later, we will take it to be the union of the ordinates and abscissa values. In practice what is typically required is the posterior distribution of \( \theta \), given the data \( d \). Assuming an uninformative prior on the parameters, \( p(\theta) = \text{constant} \), Bayes’ Theorem implies \( p(\theta|d) \propto p(d|\theta) = L \), the likelihood. The log-likelihood is then Taylor-expanded about its maximum. The first term is a constant, irrelevant for the discussion of parameter constraint forecasts; the second term is the first derivative, which vanishes at the point of maximum likelihood; the third term is the Hessian (curvature matrix) of the likelihood, and is the term whose ensemble average (over the data) gives the Fisher Matrix:

\[
F_{\alpha\beta} = -\left\langle \frac{\partial^2 \ln L}{\partial \theta_\alpha \partial \theta_\beta} \right\rangle,
\]

where \( \alpha \) and \( \beta \) label the parameters. For the case of a gaussian likelihood, this
is analytically computable, and can depend only on the expectation values of the data, $\mu(\theta) \equiv \langle d(\theta) \rangle$, and the covariance, $C(\theta) \equiv \langle (d - \mu)^T(d - \mu) \rangle$. This results in the following form for the Fisher Matrix (Tegmark, Taylor, & Heavens 1997).

$$\mathbf{F}_{\alpha\beta} = \frac{1}{2} \text{Tr} \left[ C_{-1,\alpha} C_{-1,\beta} + C_{-1}(\mu,\alpha) \mu_{T,\alpha}^{\beta} + \mu_{,\beta} \mu_{T,\alpha} \right].$$

(2)

An early example of dealing with errors in both variables was straight-line fitting, where both the statistics and astronomy communities forms used either \textit{ad hoc} choices for the axis, or ultimately arbitrary combinations e.g., the bisector or the average of the one-dimensional fits on either axis. The evolution to two-dimensional or joint-distribution fitting was accompanied by a slow transition to the Bayesian perspective (Gull, 1989). New tools for fitting data in the presence of two-dimensional errors have been developed and used to extract improved cosmological constraints from supernovae populations (March et al., 2011). Here, we develop the application of two-dimensional errors in the predictive Fisher Matrix formalism itself. For pedagogical discussions of straight-line fitting and Bayesian approaches to fitting, see for example Hogg et al. (2010); D’Agostini (2005); Kelly (2011).

The remainder of the paper is organized as follows: \section{2} describes the formal derivation of the error-bar extension to two dimensions; \section{3} describe an application of this formalism to a particular experiment, with tests on simulated data. We present conclusions in \section{4}.

\section{2. Formalism of the Extension}

Throughout this paper, we follow the formalism and notation of Bassett et al. (2009). In this method, we use a Taylor expansion of the log-likelihood, and derive the generalised Fisher Matrix from first principles.

The general aim is to find an expression for the Fisher Matrix for an experiment with gaussian errors in $X$ and $Y$, arbitrary correlations of errors (i.e. errors in $Y_i$ can be correlated with errors in $X_j$, even if $i \neq j$).
2.1. General Method with $X$-$Y$ Covariance

Let the set of measurements be the pairs $\{X_i\}, \{Y_i\}$, with $i = 1, \ldots N$. We seek the posterior probability of a set of parameters, represented collectively by $\theta$, being a vector $\theta_\alpha$ with $\alpha = 1, \ldots M$.

We assume $X_i$ and $Y_i$ have Gaussian errors, around true values $x_i$, $y_i$, with a covariance matrix $C$. $x_i$ and $y_i$ are not directly observed. This amounts to a hierarchical model, where the observables $X_i, Y_i$ depend on some unobservable latent variables $x_i$, which are essentially nuisance parameters. The $y_i$ are not independent nuisance parameters as they are assumed to be related precisely by a theoretical model $y_i = \mu(x_i)$, which depends on $\theta$. We seek the posterior $p(\theta|X,Y)$. With a uniform prior for $\theta$, this is proportional to the likelihood $L = p(X,Y|\theta)$. We write this as the marginalised distribution over $x$ and $y$ as

$$L = \int p(X,Y,x,y|\theta) \, dx \, dy = \int p(X,Y|x,y,\theta)p(x,y|\theta) \, dx \, dy \quad (3)$$

where we have expanded the condition to include the latent variables, and then further expanded the condition of $p(y)$ to include $x$.

We integrate over $y$ using a delta function, $p(y|x,\theta) = \delta(y - \mu(x))$, and assume for now a uniform prior for $x$:

$$L = \int p(X,Y|x,\mu(x),\theta) \, d^N x \quad (4)$$

At the cost of some algebraic complexity, we can introduce an informative prior (parent distribution) for $x$. In Appendix B, we generalise the analysis by assuming a gaussian population prior $p(x)$, and show that we recover the simpler result obtained in the main text in the limit that the errors in $x$ are small enough that the prior can be considered constant across the error range of individual data points. See [Gull (1989)] and [Kelly (2011)] for further discussion of this point.

Next, we make the critical assumption that we can truncate at the linear term of the Taylor expansion of $\mu$:

$$\mu(x_i) = \mu(X_i) + (x_i - X_i)\mu'(X_i); \quad (5)$$
we are essentially assuming that the function $\mu(x)$ is linear across the width of the gaussian error distribution of $x$, and this allows the likelihood to be integrated analytically.

$$ L \propto \int \frac{1}{\sqrt{\det C}} \exp \left( -\frac{Q}{2} \right) d^N x $$

where $Q \equiv (Z - z)^T C^{-1} (Z - z)$, and $z$ and $Z$ are $2N$-dimensional vectors: $z_i = x_i$ and $Z_i = X_i$ for $i \leq N$, $Z_i = Y_i$ and $z_i = \mu(X_i) + \mu'(X_i)(x_i - X_i)$ for $i > N$.

The covariance matrix of the data can be written in block form as

$$ C = \begin{pmatrix} X & Y \\ X^T & C_{XX} & C_{XY} \\ Y & C_{YX} & C_{YY} \end{pmatrix} $$

Note that $C_{XY}$ is not symmetrical, nor invertible in general; although $C_{XX}$ and $C_{YY}$ are. The inverse of $C$ is

$$ C^{-1} = \begin{pmatrix} G & -H \\ -H^T & E \end{pmatrix} $$

where

$$ G = C_{XX}^{-1} + C_{XX}^{-1} C_{XY} E C_{XY}^T C_{XX}^{-1} $$

$$ H = C_{XX}^{-1} C_{XY} E $$

$$ E = (C_{YY} - C_{XY}^T C_{XX}^{-1} C_{XY})^{-1}. $$

Defining $\tilde{x} \equiv x - X$, and $\tilde{Y} \equiv Y - \mu(X)$, we collect together the terms as follows:

$$ Q = \tilde{x}^T G \tilde{x} + (\tilde{Y} - T \tilde{x})^T E (\tilde{Y} - T \tilde{x}) \tilde{x}^T H (\tilde{Y} - T \tilde{x}) - (\tilde{Y} - T \tilde{x})^T H^T \tilde{x}, $$

where

$$ T = \text{diag}(\mu'), $$

and $\mu' = d\mu/dx|_{x=X}$, and $Q$ has the quadratic form

$$ Q = \tilde{x}^T A \tilde{x} - 2B^T \tilde{x} + Q', $$
where

\[
A = G + TET + HT + TH^T
\]
\[
B^T = Y^T(ET + H^T)
\]
\[
Q' = Y^T E \tilde{Y}.
\]  

(13)

With the definition of \(Q\) in Eqn. 12, the gaussian integral of Eqn. 6 can be performed, using

\[
\int e^{-\frac{1}{2} \hat{x}^T A \hat{x} + \hat{x}^T B \hat{\bar{x}}} dN \hat{x} = \frac{(2\pi)^{N/2}}{\sqrt{\det A}} e^{\frac{1}{2} B^T A^{-1} B},
\]  

(14)

and noting that \(Q'\) is independent of \(\hat{x}\). The likelihood then simplifies after a few lines of algebra to

\[
L \propto \frac{1}{\sqrt{\det A \det C}} \exp \left( - \frac{1}{2} \tilde{Y}^T R^{-1} \tilde{Y} \right),
\]  

(15)

where the inverse of the marginal covariance matrix of \(\tilde{Y}\) is

\[
R^{-1} = E - (ET + EC_{XY} C_{XX}^{-1}) A^{-1} (TE^T + C_{XX}^{-1} C_{XY} E).
\]  

(16)

Next, we obtain the marginal covariance \(R\) in terms of the covariances \(C_{ii}\). We first expand \(A\) and simplify it to

\[
A = C_{XX}^{-1} + (C_{XX}^{-1} C_{XY} + T) E (C_{XX}^{-1} C_{XY} + T)^T,
\]  

(17)

and use the Woodbury formula [Woodbury, 1950]

\[
(K + UWV)^{-1} = K^{-1} - K^{-1} U (W^{-1} + VK^{-1} U)^{-1} VK^{-1}
\]  

(18)

to obtain the inverse,

\[
A^{-1} = 
C_{XX} - C_{XX} (C_{XX}^{-1} C_{XY} + T) (E^{-1} + [C_{XX}^{-1} C_{XY} + T]^T \times 
C_{XX} [C_{XX}^{-1} C_{XY} + T]^{-1} (C_{XX}^{-1} C_{XY} + T)^T C_{XX}.
\]  

(19)

Substituting this into \(R\) and simplifying, we find

\[
R = C_{YY} + C_{XY}^T T + TC_{XY} + TC_{XX} T,
\]  

(20)
which is the key result of the calculation. We can also simplify the pre-factor, \( \det A \det C = \det R \) (see Appendix A for the proof). Thus

\[
L \propto \frac{1}{\sqrt{\det R}} \exp \left( -\frac{1}{2} \tilde{Y}^T R^{-1} \tilde{Y} \right).
\]  

(21)

We see that this looks just like a normal gaussian (in terms of data) likelihood, but with the covariance matrix \( C \) (\( C_{YY} \) in our current notation) replaced by \( R \). Hence to compute the Fisher matrix, we can use the standard formula found in Eqn. 2 and Eqn. 15 of Tegmark, Taylor, & Heavens (1997), and simply replace \( C \) by \( R \):

\[
F_{\alpha\beta} = \frac{1}{2} \text{Tr} \left[ R^{-1} R_{,\alpha} R^{-1} R_{,\beta} + R^{-1} (\mu_{,\alpha} \mu_{,\beta}^T + \mu_{,\beta} \mu_{,\alpha}^T) \right].
\]  

(22)

This is the main result of this paper. Note that \( R \) depends not only on the standard covariance, but also on the covariance in the independent variable, \( C_{XX} \), the meta-covariance, \( C_{XY} \), and the first derivative of the model function \( \mu \). In the case of uncorrelated data pairs, the result reduces to that found in March et al (2011). For the simple case of no correlations between \( X \) and \( Y \) values \( R = C_{YY} + TC_{XX}T \), and with diagonal covariance matrices \( C_{YY} \) and \( C_{XX} \) we recover the propagation of error result that the variance of \( f \equiv Y - \mu(X) \) for each data point is effectively

\[
\sigma_f^2 = \sigma_Y^2 + \mu'(X)^2 \sigma_X^2,
\]  

(23)

with an obvious notation, and \( C \) can be replaced in the standard Fisher expression by a diagonal \( N \times N \) matrix with these enhanced entries.

We now briefly make a few key observations. First, when the derivatives of the model function are zero (\( T = 0 \)), then the latent variable \( x \) has no bearing on \( R \), and we recover the usual formula for the Fisher Matrix: when \( T = 0 \), \( R = C_{YY} \). Also, in the limit of infinitesimal errors in \( X \), we recover the usual Fisher matrix formula. As remarked earlier, if the errors in \( X \) and \( Y \) are uncorrelated, and in the limit that the errors in \( X \) are small in comparison with the width of the prior, we recover the result obtained from propagation of errors, namely that the variance of \( Y \) is effectively increased from \( \sigma_Y^2 \) to \( \sigma_Y^2 + \mu'^2 \sigma_X^2 \). Also, although the main focus of the paper has been on the Fisher matrix, the expression for the likelihood itself
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(equation 21) can be used without the usual interpretation that it is gaussian in the parameter space, to make predictions for the shape of the likelihood surfaces beyond ellipses. Thus the technology of DALI (Sellentin et al., 2014) can be generalized straightforwardly by replacing the data covariance matrix by \( R \). Finally, even if the covariance matrix of the data (the original \( C \), which is \( C_{YY} \)) is independent of the parameters, \( R \) is not, because in general \( \mu' \) (and hence \( T \)) do depend on the parameters.

3. Example Application

A classic example of inference in cosmology comes from the Type 1A supernova Hubble diagram, which consists of data pairs corresponding to the redshift of the host galaxy of each supernova, and its apparent brightness. Various corrections, including one based on colour, are applied such that these supernovae act as standard candles with a small dispersion of around 10%. For relatively nearby supernovae, the redshift measures recession velocity, the apparent brightness is a measure of distance, and the relation is linear (Hubble’s law). At large distances, the effects of curvature of the Universe and finite light travel times make the relation deviate from linear, and observations can be used to infer the acceleration of the Universe, or, in the context of the Lambda Cold Dark Matter (ΛCDM) model, the matter and dark energy content. See March et al. (2011) for a principled analysis of data and further discussion of the background. Redshifts obtained from spectroscopy are negligibly small, but if they are photometric redshifts, based on broad-band colours of the host galaxy, then two complications arise. One is that the redshift errors may be large (typically around 5 or 10% for 5-band photometry). The second is that errors in the photometry (such as zero-point errors) will introduce errors in the redshifts, but could also affect the colour corrections for the supernovae themselves. This potentially couples the errors in \( X \) and \( Y \) for a given data pair. Kim & Miguel (2007) investigated correlations between redshift and magnitude errors in photometric surveys, and found rather variable correlation coefficients between
**Fig. 1.** A scenario in which the formation of overlapping weighted combinations of the original data may lead to correlations between $X$ and $Y$ values of different pairs. Here, the $Y$ values have been adjusted to the theoretical curve for a fiducial set of model parameters, which is a function of $X$, so errors in $X$ propagate into $Y$, and the weighting then mixes different $Y$ values. This then correlates both $X$ and $Y$ values from different pairs. $\mu$ and $\mu_{\Lambda CDM}$ are the measured and theoretical distance moduli, with the theoretical model chosen for illustration to be the $\Lambda CDM$ concordance model.

A scenario which could couple the errors in $X$ and $Y$ for different data pairs arises if one takes weighted averages of the data. This one might do in order to make the errors closer to gaussian, as we do not know the error distribution for individual supernovae. If this is done with overlapping sub-samples, to maintain a good sampling in redshift (see Fig. 1), then the errors will be coupled. Furthermore, if the $Y$ values are referred to a fiducial model (such as the standard cosmological model), as shown, then this involves dividing by a function of the supernova redshift, which then couples the errors in $X$ to the errors in $Y$ across different (weighted) data pairs. So we see in this example how one can get full covariance between $X$ and $Y$ sets, with non-zero off-diagonal terms of all types.

In Fig. 2 we see the results of a simulation of a supernova Hubble diagram.

![Diagram](image-url)
Fig. 2. Generalised Fisher Matrix calculations compared with MCMC results from simulated supernova data generated with correlations between $X$ and $Y$ values in each data pair. The likelihood is accurately a bivariate gaussian for this example, and there is good agreement in the shape, size and orientation of the ellipses, with the actual likelihood offset from the true solution in accordance with expectation.

with correlated errors, where we obtain an estimate of the posterior for the matter density and Hubble constant using Markov Chain Monte Carlo techniques. For this case, the posterior (with uniform priors on the parameters) is closely described by a bivariate gaussian, and comparison with the expected error contours from the generalised Fisher Matrix technique, also shown, gives excellent agreement.

4. Conclusions

In this paper we have considered the Fisher Information Matrix for data which have gaussian errors in both variables of each data pair. These errors can have
arbitrary correlations between data points, such as correlations between one independent variable and a different independent variable. The main result, equation [22], is similar to the standard Fisher matrix, but with the covariance matrix replaced by a more complicated matrix derived from the expanded covariance matrix of all variables, and the derivative of the mean signal with respect to the dependent variable. The result is valid for situations where two conditions hold: the first is that a Taylor expansion of the mean signal to linear order is valid across the gaussian error of the independent variable; the second is that the errors in the independent variable are small compared with the width of the prior distribution. At the price of some complexity, we present a perturbative correction when the latter condition does not hold. In the case when the errors are uncorrelated between data pairs, the result reduces to the result one obtains from propagation of errors, where the variance of the dependent variable is increased from $\sigma_Y^2$ to $\sigma_Y^2 + \mu^2 \sigma_X^2$. Since we compute the likelihood itself, it may be used to evaluate the expected likelihood surface when it is not gaussian in the parameter space, straightforwardly generalizing the DALI technique of [Sellentin et al. (2014)]. Finally, the generalised Fisher Matrix has been implemented in the Fisher4Cast software, available at [http://www.mathworks.com/matlabcentral/fileexchange/20008-fisher-matrix-toolbox-fisher4cast](http://www.mathworks.com/matlabcentral/fileexchange/20008-fisher-matrix-toolbox-fisher4cast).

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**A. Proof that** $\det C \det A = \det R$

With $\det C = \det(C_{XX}) \det(C_{YY} - C_{XY}^{-1} C_{XY} C_{XX}^{-1} C_{XY}) = \det(C_{XX}) \det(E^{-1})$, we have, reversing the order of the determinants,

$$\det C \det A = \det(E^{-1}) \det(C_{XX}) \times \det \left[ C_{XX}^{-1} + (C_{XX}^{-1} C_{XY} + T) E (C_{XX}^{-1} C_{XY} + T)^T \right]$$

(24)
Now, since \( \det U \det V = \det(UV) \) for any square matrices,

\[
\det C \det A = \det(E^{-1}) \det \left[ I + (C_{XY} + C_{XX} T) E(C_{XX}^{-1} C_{XY} + T)^T \right].
\]  

(25)

Now we use Sylvester’s Determinant Theorem, \( \det(I + UV) = \det(I + VU) \) where we take \( V = E(C_{XX}^{-1} C_{XY} + T)^T \):

\[
\det C \det A = \det(E^{-1}) \det \left[ I + E(C_{XX}^{-1} C_{XY} + T)^T (C_{XY} + C_{XX} T) \right].
\]  

(26)

Using \( \det U \det V = \det(UV) \) again, and expanding \( E^{-1} \),

\[
\det C \det A = \det \left[ E^{-1} + (C_{XX}^{-1} C_{XY} + T)^T (C_{XY} + C_{XX} T) \right]
= \det \left[ C_{YY} - C_{XY} T C_{XX}^{-1} C_{XY} + (C_{XY} T C_{XX}^{-1} + T) (C_{XY} + C_{XX} T) \right]
= \det \left[ C_{YY} + C_{XY} T + T C_{XY} + T C_{XX} T \right] = \det R.
\]  

(27)

### B. Generalisation to non-uniform prior, or parent distribution

We now generalise the method to apply to cases where the prior in \( x \) is not uniform. We illustrate this with a simplifying assumption that the prior is a gaussian of specified width, and demonstrate that in the limit of a prior width which is much larger than the errors in \( x \), we recover the results in the main text, and we expect this to hold for any broad prior. We can consider a prior which is dependent on each point, with a mean vector \( a \) and variance \( \Sigma \) (we assume that \( \Sigma \) is a diagonal matrix).

In the normal case where the abscissa values are drawn from the same distribution, then all elements of \( a \) are identical, and \( \Sigma \) is proportional to the identity matrix.

Assuming a gaussian prior

\[
p(x) \propto \exp\left[ -\frac{1}{2} (x - a)^T \Sigma^{-1} (x - a) \right]
\]  

(28)

we get for the likelihood

\[
L \propto \int \frac{1}{\sqrt{\det C}} \exp \left\{ -\frac{1}{2} \left[ Q + (x - a)^T \Sigma^{-1} (x - a) \right] \right\} d^N x,
\]  

(29)

Subsequently, we get

\[
Q + (x - a)^T \Sigma^{-1} (x - a) = \hat{X}^T \hat{A} \hat{x} - 2 \hat{B}^T \hat{x} + \hat{X}^T \Sigma^{-1} \hat{X} + Q'
\]  

(30)
where
\[ \tilde{A} = A + \Sigma^{-1} \tag{31} \]
\[ \tilde{B}^T = B^T - \tilde{X}^T \Sigma^{-1}. \tag{32} \]

We perform the gaussian integral as before, finding
\[ L \propto \frac{1}{\sqrt{\det C \det \tilde{A}}} \exp \left[ -\frac{1}{2} \left( -\tilde{B}^T \tilde{A}^{-1} \tilde{B} + \tilde{X}^T \Sigma^{-1} \tilde{X} + Q' \right) \right]. \tag{33} \]

In the case when the prior in \( x \) is informative, then there is information in the values of \( X \), so the data vector should include both \( X \) and \( Y \). The likelihood is then
\[ L \propto \frac{1}{\sqrt{\det C \det \tilde{A}}} \exp \left( -\frac{1}{2} Q_{YX} \right) \tag{34} \]
where
\[ Q_{YX} = (\tilde{Y}, \tilde{X})^T J(\tilde{Y}, \tilde{X}) \tag{35} \]

and, collecting terms and using the Woodbury identity again, we find
\[ J = \begin{pmatrix} E - (ET + H^T)(A + \Sigma^{-1})^{-1}(TE^T + H) & (ET + H^T)(A + \Sigma^{-1})^{-1} \Sigma^{-1} \\ \Sigma^{-1}(A + \Sigma^{-1})^{-1}(TE^T + H) & (\Sigma + A^{-1})^{-1} \end{pmatrix}. \tag{36} \]

In the limit of an infinitely broad prior, we see that, as expected, \( \tilde{X} \) contains no useful information, and the likelihood depends only on \( \tilde{Y} \), with the quadratic simplifying to \( Q_{YX} \to Q_Y \equiv \tilde{Y}^T R \tilde{Y} \), and as expected, we recover the results of the main text.

To investigate departures from the main text result, we consider terms linear in \( \Sigma^{-1} A^{-1} \). This approximation only makes sense if
\[ \lim_{n \to \infty} (\Sigma^{-1} A^{-1})^n = 0. \tag{37} \]

As \( \Sigma \) is a diagonal matrix, the elements of the matrix \((\Sigma^{-1} A^{-1})^n\) are given by
\[ \left[ (\Sigma^{-1} A^{-1})^n \right]_{ij} = \left( [\Sigma^{-1}]_{ii} [A^{-1}]_{ii} \right)^{n-1} [\Sigma^{-1}]_{ii} [A^{-1}]_{ij} \]
\[ = \left( [A^{-1}]_{ii} / \Sigma_{ii} \right)^{n-1} [A^{-1}]_{ij} / \Sigma_{ii} \tag{38} \]
Thus condition \([A^{-1}]_{ii} \ll \Sigma_{ii}\) is fulfilled if

\[
[A^{-1}]_{ii} \ll \Sigma_{ii}
\]  

(39)

for all \(i\). We will assume this and neglect higher order terms in \(\Sigma^{-1}A^{-1}\). Then we can approximate \(\tilde{A}^{-1}\) by

\[
\tilde{A}^{-1} = (A + \Sigma^{-1})^{-1}
\]

(40)

\[
= A^{-1}(1 + \Sigma^{-1}A^{-1})^{-1} \simeq A^{-1}(1 - \Sigma^{-1}A^{-1})
\]

Inserting this result in equation (33), we get

\[
L \propto L_0L_1
\]

(41)

with

\[
L_0 = \frac{1}{\sqrt{\det C \det A}} \exp \left( -\frac{1}{2} \tilde{Y}^T R^{-1} \tilde{Y} \right)
\]

(42)

and

\[
L_1 = \frac{1}{\sqrt{\det (I + \Sigma^{-1}A^{-1})}} \exp \left[ -\frac{1}{2} (A^{-1}B + X - a)^T \Sigma^{-1} (A^{-1}B + X - a) \right].
\]

(43)

\(L_0\) is the zeroth order result from the main text.

The Fisher matrix is then given by

\[
F_{\alpha\beta} = F_{\alpha\beta}^{(0)} + F_{\alpha\beta}^{(1)}
\]

(44)

with

\[
F_{\alpha\beta}^{(i)} = -\left\langle \frac{\partial^2 \ln L_i}{\partial \theta_\alpha \partial \theta_\beta} \right\rangle \quad i = 0, 1.
\]

(45)

We already know the result for \(F_{\alpha\beta}^{(0)}\), so we just need to calculate the first-order term:

\[
F_{\alpha\beta}^{(1)} = \left\langle \frac{\partial^2}{\partial \theta_\alpha \partial \theta_\beta} \left[ \frac{1}{2} \ln \det (I + \Sigma^{-1}A^{-1}) + \frac{1}{2} (A^{-1}B + X - a)^T \Sigma^{-1} (A^{-1}B + X - a) \right] \right\rangle.
\]

(46)

Using the approximation

\[
\ln \det (I + \Sigma^{-1}A^{-1}) = \text{Tr} \ln (I + \Sigma^{-1}A^{-1}) \simeq \text{Tr} (\Sigma^{-1}A^{-1})
\]

(47)
and with $\langle \tilde{Y} \rangle = 0$, $\langle \tilde{Y} \tilde{Y}^T \rangle = R$, and $\tilde{Y}_{,\alpha} = -\mu_{,\alpha}$ we find after some tedious calculations

$$F_{\alpha\beta}^{(1)} = \frac{1}{2} \text{Tr} \left[ \Sigma^{-1} \{ A^{-1} \}_{,\alpha\beta} + \{ (E^T + H^T) A^{-1} \Sigma^{-1} A^{-1} (TE + H) \}_{,\alpha\beta} R \right]$$

$$- (X - a)^T \Sigma^{-1} \{ A^{-1} (TE + H) \mu \}_{,\alpha\beta} + (X - a)^T \Sigma^{-1} \{ A^{-1} (TE + H) \}_{,\alpha\beta} \mu + \mu_{,\alpha}^T (E^T + H^T) A^{-1} \Sigma^{-1} A^{-1} (TE + H) \mu_{,\beta}.$$ (48)

As $\{ A^{-1} \}_{,\alpha} = -A^{-1} A_{,\alpha} A^{-1}$, each term in (48) contains the factor $\Sigma^{-1} A^{-1}$, so $F_{\alpha\beta}^{(1)}$ gives the first-order corrections in terms of this parameter.

References


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