Optimal PSF modelling for weak lensing: complexity and sparsity

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ABSTRACT

Context. Controlling shape measurement systematics in weak gravitational lensing.
Aims. Quantifying the effect of systematic errors in modelling the Point Spread Function (PSF) on cosmological parameter measurements from cosmic shear.
Methods. We explore the impact of PSF fitting errors on cosmic shear measurements using the concepts of complexity and sparsity. Complexity, introduced in a previous paper, characterises the number of degrees of freedom of the PSF. For instance, fitting an underlying PSF with a model with low complexity will lead to small statistical errors on the model parameters, however these parameters could suffer from large biases. Alternatively, fitting with a large number of parameters will tend to reduce biases at the expense of statistical errors. We perform an optimisation of scatters and biases by studying the Mean Squared Error (MSE) of a PSF model. We also characterise a model sparsity, which describes how efficiently the model is able to represent the underlying PSF using a limited number of free parameters. We present the general case and illustrate it for a realistic example of a PSF fitted with the shapelet basis set.
Results. We derive the relation between complexity and sparsity of the PSF model, Signal-to-Noise Ratio of stars and systematic errors on cosmological parameters. With the constraint of maintaining the systematics below the statistical uncertainties, this lead to a relation between the required number of stars to calibrate the PSF and the sparsity. We discuss the impact of our results for current and future cosmic shear surveys. In the typical case where the biases can be represented as a power law of the complexity, we show that current ground surveys can calibrate the PSF with few stars, while future surveys will require hard constraints on the sparsity in order to calibrate the PSF with 50 stars.

Key words. Gravitational lensing - Cosmology: dark matter - Cosmology: cosmological parameters

1. Introduction

Studying spatial correlations between galaxy shapes induced by gravitational lensing of the large scale structure (‘Cosmic Shear’) is a powerful probe of dark energy and dark matter. A number of current and planned surveys are dedicated to cosmic shear, such as: the Canada-France-Hawaii-Telescope Legacy Survey\textsuperscript{4} (CFHTLS), the Kilo Degree Survey and the VISTA Kilo-Degree Infrared Galaxy Survey\textsuperscript{5} (KIDS/VIKING), the Dark Energy Survey\textsuperscript{6} (DES), the Panoramic Survey Telescope & Rapid Response System\textsuperscript{4} (Pan-STARRS), the SuperNovae Acceleration Probe\textsuperscript{5} (SNAP), the Large Synoptic Survey Telescope\textsuperscript{6} (LSST) and the Dark UNiverse Explorer\textsuperscript{7} (DUNE/Euclid).

The most efficient way to improve the statistical precision of cosmic shear analyses is to enlarge the surveys. Amara & Refregier (2007a) show that, once the median redshift is sufficiently high ($z \gtrsim 0.7$), it is always advantageous to make cosmic shear surveys as wide as possible, rather than deep, in order to minimise the error bars on cosmological parameters. Currently, the largest

\begin{enumerate}
\item http://www.cfht.hawaii.edu/Science/CFHLS/
\item http://www.eso.org/sci/observing/policies/
PublicSurveys/sciencePublicSurveys.html
\item http://www.darkenergysurvey.org
\item http://pan-starrs.ifa.hawaii.edu
\item http://snap.lbl.gov
\item http://www.lsst.org
\item http://www.dune-mission.net
\item http://www.esa.int/esaCP/index.html
\end{enumerate}
data set optimised for cosmic shear is the Wide field of the CFHTLS, which covers 50 deg$^2$ \cite{2008Natur.455...61F} and will eventually reach 170 deg$^2$. Another analysis has also been published that combines 4 surveys which together cover an area of 100 deg$^2$ \cite{2007MNRAS.377..507B}. In a few years, the KIDS/VIKING survey will cover 1500 deg$^2$. Eventually, projects that are currently being planned, such as DUNE/Euclid, which is planned for 2017, will be able to perform cosmic shear measurements over the entire observable extra-galactic sky ($\sim 20,000$ deg$^2$).

In order for cosmic shear surveys to reach their full potential, it is necessary to ensure that systematic errors are sub-dominant relative to statistical uncertainties. In particular, a tight control of all the effects associated with shape measurements is required. To illustrate the difficulty of making accurate shear measurements, we begin with an overview of the ‘forward process’ that shows the way that a galaxies original image is distorted to form the final image that we measure. In the forward process a galaxy image is: (i) sheared by gravitational lensing; (ii) convolved with a Point Spread Function (PSF) that comes from a number of sources (instruments, atmosphere, . . . ); (iii) pixelated at the detector; and finally (iv) subject to noise.

Cosmic shear analyses involve the reverse process: we begin with the final image and we move backwards from step (iv) to (i) in order to recover the original lensing effect. A detailed and illustrated description of the forward and inverse processes is given in the GREAT08 Challenge Handbook \cite{2008ASPC..392..737B}. The GREAT08 Challenge aims to bring a wide range of expertise into gravitational lensing by presenting the relevant issues in a clear way so as to make it accessible to a broad range of disciplines, including the machine-learning and signal-processing communities. Other similar challenges have also been performed within the weak lensing community, as part of the STEP collaboration \cite{2006NewA...11..259H,2007ASPC..376..289M,2008ASPC..392..715R}, which focussed mainly on understanding the systematics at play in current shear measurement methods. These challenges focus mainly on reducing the errors which stem from the shape measurement method. However, even with a perfect method there are underlying limits that come from the fact that each measurement (of either stars and galaxies) contain a finite amount of information.

In \cite{2008NewA...13S.139P} (P1 hereafter), we investigate the link between systematic errors in the power spectrum and uncertainties at the PSF correction phase. The framework is the following: because the PSF of an instrument will vary on all scales, the PSF needs to be measured using the stars that surround the lensed galaxy. Each star provides an image of the PSF that is pixelated and noisy, which means that to reach a given accuracy on the knowledge of the PSF, a number of stars is required. We estimate in P1 the minimum number of stars \( N_s \) required to calibrate the PSF with a given accuracy, according to the stellar Signal-to-Noise Ratio (SNR), the minimum galaxy size, the complexity of the PSF and the tolerated variance of the systematics \( \sigma^2_{\text{sys}} \). On the other hand, \cite{2007MNRAS.377..637A} give the upper limit that can be tolerated on \( \sigma^2_{\text{sys}} \) when estimating cosmological parameters. Combining both papers together allow one to derive the minimum number of stars required to reach a given accuracy. For instance, current results, that allow us to constrain \( \sigma^2 \Omega_M \) with an accuracy of 0.05, require \( \sigma^2_{\text{sys}} \) lower than a few \( 10^{-6} \) and the PSF to be calibrated with 5 stars; while for future ambitious surveys that will allow us to constrain \( w_0 \) and \( w_a \) with an accuracy of 0.02 and 0.1 respectively, \( \sigma^2_{\text{sys}} \) must be lower than \( 10^{-7} \), requiring at least 50 stars (for stars with signal-to-noise ratio of 500 and a PSF described with few degrees of freedom, as can be typically achieved in space).

In P1 we use the same functional form for both the underlying PSF as well as the model used to fit it. This means that the PSF model is able to perfectly describe the underlying PSF. Errors in the fit due to the noise causes a scatter of the fitted parameters around the truth. For instance, if the model is an orthogonal basis set, then the fitted parameters follow a Gaussian distribution around the truth. In this paper we extend this investigation by studying the impact of fitting a PSF with a model that has a different form. This addresses the case where the underlying PSF, which is unknown in practice, is estimated by fitting the parameters of an arbitrary model. This can lead to both a scatter in the fitting parameters and an offset of the average relative to the truth, i.e. a bias in the fitting parameters. This leads to an important optimisation since when presented with a given PSF we have the choice to fit using a complex model, which would have a small bias but a large scatter, or a simpler model that would have a smaller scatter but a larger bias. To quantify these effects we will revisit the notion of complexity proposed in P1 and introduce the concept of sparsity.

This paper is organised as follow: first we discuss in section 2 the concepts of complexity and sparsity, which are the key concepts of this paper; section 3 presents the notation; section 4 present the definition of optimal complexity, illustrates our formalism with a PSF example and uses the sparsity as a tool for optimising the complexity; section 5 derives the minimum number of stars required to calibrate the PSF, extending results of P1; and finally section 6 summarises our conclusions.

2. Complexity and Sparsity

In P1 we introduce the concept of complexity: we show that few complexity factors characterise the amount of information that needs to be collected about the PSF. This is summarised and revisited in section 2.1. In section 2.2 we introduce the concept of sparsity which is a measure of how a PSF model is efficient to represent the underlying PSF with a small number of free parameters. This therefore allows us to explore how an optimal PSF model can be constructed to minimise \( \sigma_{\text{sys}} \).
2.1. Complexity

We define in P1 the complexity factors of the PSF, that characterise the number of Degrees of Freedom (DoF) that are estimated from stars (in the limit of infinite resolution, i.e. infinitely small pixels): the larger the number of DoF, the larger the complexity factors. Each PSF shape parameter is associated with a complexity factor related to the rms of its estimator. In the simple formalism where we consider unweighted quadrupole moments, the PSF is characterised by only two complexity factors \( \psi_1 \) and \( \psi_2 \) associated with the 2 component PSF ellipticity \( \epsilon_{PSF} \) and the square PSF rms radius \( R_{PSF}^2 \) respectively (as defined in P1). For a given star, one has:

\[
\left( \psi_{R^2}, \psi_\epsilon \right) \equiv S_\star \left( \frac{\sigma[R_{PSF}^2]/R_{PSF}^2}{\sigma[\epsilon_{PSF}]} \right)
\]

where \( S_\star \) is the photometric Signal-to-Noise Ratio (SNR) of the star and \( \sigma[R_{PSF}^2], \sigma[\epsilon_{PSF}] \) are the rms errors of the PSF size and ellipticity estimators respectively. As in P1, we assume the small ellipticity regime (i.e. \( |\epsilon_{PSF}| \lesssim 0.1 \)) implying that the measure of \( \epsilon_{PSF} \) is isotropic and the 2 components of the ellipticity have the same rms uncertainty, i.e. \( \sigma[\epsilon_{PSF},i] \equiv \sigma[\epsilon_{PSF}] \).

In the case where the PSF can be considered constant over several stars, or for particular representations of the PSF (for example with shapelet basis sets in the small ellipticity regime, see P1), \( \psi_{R^2} \) and \( \psi_\epsilon \) are spatially constant and equation \( \Box \) can be extended to a set of several stars. For a combination of several stars, \( S_\star \) becomes \( \sqrt{n_s} S_{\text{eff}} \):

\[
\left( \psi_{R^2}, \psi_\epsilon \right) \equiv \sqrt{n_s} S_{\text{eff}} \left( \frac{\sigma[R_{PSF}^2]/R_{PSF}^2}{\sigma[\epsilon_{PSF}]} \right),
\]

where \( S_{\text{eff}} \) is the effective stellar SNR and \( n_s \) the effective number of stars \( k \) used for the PSF calibration, as defined by (see P1):

\[
n_s S_{\text{eff}}^2 \equiv \sum_k S_k^2.
\]

We also show in P1 that the polar shapelet basis set, proposed by Massey & Refregier (2005), tested on simulated data in Massey et al. (2007) and used on real data by Bergé et al. (2008), is particularly convenient to model the PSF: in the small ellipticity regime, \( \psi_1 \) and \( \psi_2 \) depend only on the polar shapelet basis set over which the PSF is decomposed (not on the PSF itself). For this reason we work with shapelets in this paper when illustrating our discussions with an example. Note that our results and conclusions are not restricted to shapelets but remain valid whatever the PSF model. For convenience, we work with the shapelet ‘diamond’ option (described in details in P1) that imposes a lower limit to the scales which are described and links \( \psi_\epsilon \) and \( \psi_{R^2} \):

\[
\psi_\epsilon^2 = \psi_{R^2}^2 - N,
\]

with \( N \) the largest even integer lower than or equal to the order \( n_{\text{max}} \) of the basis set. We then consider the overall complexity \( \Psi \) defined in the following according to \( \psi_\epsilon, \psi_{R^2} \) and the variance of the galaxy ellipticity distribution.

2.2. Sparsity

In this paper we introduce the concept of sparsity of the PSF model, which describes how efficiently a model is able to represent the underlying PSF with a limited number of DoF (i.e. with a limited complexity). Specifically, the sparsity quantifies how the residuals between the estimated and the underlying PSF decrease when the complexity of the PSF model increases. With a large number of DoF, i.e. a high complexity, one might expect small residuals but large scatter of the fitted parameters. On the other side, with a small number of DoF, i.e. a low complexity, one might expect large residuals but small scatter of the fitted parameters. The sparsity characterises the slope of this relation and thus is an estimate of how much information can be contained in a given number of DoF. We show how to use the sparsity for optimising the complexity in order to minimise \( \sigma_{\text{sys}} \).

Consider the shape parameters \( R_{PSF}^2 \) and \( \epsilon_{PSF} \) of the underlying PSF, as defined previously. The differences \( \delta(R_{PSF}^2) \) and \( \delta \epsilon_{PSF} \) between the underlying PSF (‘true’ index) and its estimation (‘est’ index) can be written:

\[
\delta(R_{PSF}^2) = R_{PSF}^2 - R_{PSF}^2 \text{ est},
\]

\[
\delta \epsilon_{PSF} = \epsilon_{PSF} - \epsilon_{PSF} \text{ est}.
\]

These differences are of two kinds: the statistical scatter relative to the average \( \sigma \) and the bias-offset \( b \) of the average relative to the truth. The Mean Square Errors (MSE) of \( R_{PSF}^2 \) and \( \epsilon_{PSF} \) are:

\[
\text{MSE}[R_{PSF}^2] = \sigma^2[R_{PSF}^2] + b^2[R_{PSF}^2],
\]

\[
\text{MSE}[\epsilon_{PSF},i] = \sigma^2[\epsilon_{PSF},i] + b^2[\epsilon_{PSF},i] \text{ with } i = 1, 2.
\]

In P1, we address the zero bias case \( b(\epsilon_{PSF}) = 0 \), considering the PSF model is able to perfectly describe the underlying PSF. However, nulling the biases is not necessarily the optimal PSF modelling. It can be advantageous to work with a simplistic PSF model that is not able to mimic all the PSF features and has some biases but a low statistical scatter (see our PSF example section \( \Box \) and figure \( \Box \)). This paper proposes another approach that consists of optimising the PSF model to minimise \( \sigma_{\text{sys}} \). We do this by searching for the optimal trade-off between the systematic errors \( b(\epsilon_{PSF}) \), \( b[R^2] \) and the statistical errors \( \sigma[\epsilon], \sigma[R^2] \), which is equivalent to searching the optimal complexity \( \Psi \) of the PSF model. The sparsity is the concept that allows us to perform this optimisation as it characterises the decrease of the biases \( b \) when \( \Psi \) increases.

In the following we define a ‘sparsity parameter’ \( \alpha \) in the particular case where the biases are modeled as a power law of the PSF model complexity (\( B \propto \Psi^{\alpha} \)), and we study the impact of \( \alpha \) on the number of stars required to calibrate the PSF. We thus revisit the main result of P1 by deriving \( N_\star \) (the number of stars required to calibrate the PSF) according to \( \alpha \) instead of \( \Psi \) (the complexity of the underlying PSF). Moreover, this new relation is optimised in order to minimise \( \sigma_{\text{sys}} \).

We underline that, in this paper, we propose the sparsity as a tool for optimising the complexity of the PSF
model within a given basis set. We do not address the issue of choosing the basis set of the PSF model. There is no doubt that, in order to optimise the PSF modelling, it is necessary to carefully choose this basis set. For instance generic basis sets such as shapelets, wavelets or Fourier modes, although there have huge advantages, are not optimal and will probably not be used in the current state in future cosmic shear analyses. But here, we do not look at the relevance of the basis set. We propose to optimise the complexity of the PSF model within a given basis set.

3. PSF calibration for shear measurement

When deconvolving the observed galaxy with the estimated PSF, $\delta \sigma_{\text{PSF}}$ and $\delta \epsilon_{\text{gal}}$ propagate into an error $\delta \epsilon_{\text{gal}}$ in the estimation of the galaxy ellipticity. Let us denote $R_{\text{gal}}$ and $\epsilon_{\text{gal}}$ as the rms radius and the two component ellipticity of the galaxy. When $R_{\text{PSF}}$, $\epsilon_{\text{PSF}}$, $R_{\text{gal}}$ and $\epsilon_{\text{gal}}$ are defined using the unweighted moments of the flux, this propagates to (see P1):

$$ \delta \epsilon_{\text{gal}} \simeq (\epsilon_{\text{gal}} - \epsilon_{\text{PSF}}) \frac{\delta (R_{\text{PSF}})}{R_{\text{gal}}} - (\frac{R_{\text{PSF}}}{R_{\text{gal}}})^2 \delta \epsilon_{\text{PSF}}. \quad (8) $$

Moreover, the spatial average of $|\delta \epsilon_{\text{gal}}|^2$ is related to the variance of the systematic errors in shear measurements $\sigma_{\text{sys}}^2$ [Amara & Refregier (2007b)] defined as the integral of the systematics $C_{\ell}^\text{sys}$ on the power spectrum:

$$ \sigma_{\text{sys}}^2 \equiv \frac{1}{2\pi} \int d(\ln \ell) \ell(\ell + 1) |C_{\ell}^\text{sys}| \quad (9) $$

$$ = (P^\gamma)^{-2} \left\langle |\delta \epsilon_{\text{sys}}|^2 \right\rangle \quad (10) $$

with $P^\gamma$ the calibration factor between the gravitational shear and the ellipticity. Its value depends on the distribution of galaxy ellipticities and is typically around 1.84 [Rhodes et al. (2000)]. The brackets $\langle \rangle$ denote a spatial average over the entire field. As in P1, we substitute equation [][ into equation [][ with the following simplifying assumptions:

1. The galaxy is not correlated with the PSF.
2. The error on the PSF ellipticity ($\delta \epsilon_{\text{PSF}}$) and the PSF ellipticity itself ($\epsilon_{\text{PSF}}$) are not correlated. This is warranted by the fact that, in the assumed small ellipticity regime, $\delta \epsilon_{\text{PSF}}$ does not have any preferred direction, implying $\langle \epsilon_{\text{PSF}} \delta \epsilon_{\text{PSF}} \rangle = 0$.
3. We neglect correlations between the ellipticity and the inverse squared radius of the galaxy. This is reasonable for this work on the PSF calibration in the small ellipticity regime.

With these simplifications, substituting equation [8] into equation [10] gives:

$$ \sigma_{\text{sys}}^2 = (P^\gamma)^{-2} \left( \frac{R_{\text{PSF}}}{R_{\text{gal}}} \right)^4 \left[ \langle |\delta \epsilon_{\text{PSF}}|^2 \rangle + \left\langle |\epsilon_{\text{gal}}|^2 \right\rangle \frac{\langle \delta R_{\text{PSF}}^2 \rangle}{R_{\text{PSF}}^4} \right]. \quad (11) $$

We make this more compact by adopting the following notation:

$$ C \equiv (P^\gamma)^{-2} \left( \frac{R_{\text{PSF}}}{R_{\text{gal}}} \right)^4, \quad (12) $$

$$ \mathcal{E} \equiv \left\langle |\epsilon_{\text{gal}}|^2 \right\rangle + \left\langle |\epsilon_{\text{PSF}}|^2 \right\rangle, \quad (13) $$

which leads to:

$$ \sigma_{\text{sys}}^2 = C \left[ \langle |\delta \epsilon_{\text{PSF}}|^2 \rangle + \frac{\mathcal{E}}{R_{\text{PSF}}^2} \langle |\delta R_{\text{PSF}}|^2 \rangle \right]. \quad (14) $$

In P1, we consider only the scatters (i.e. zero bias case: $b(\epsilon_i) = b[R^2] = 0$) and we approximate the statistical averages with spatial averages: $\sigma^2[R_{\text{PSF}}^2] \simeq |\delta R_{\text{PSF}}|^2$ and $\sigma^2[\epsilon_{\text{PSF}},i] \simeq |\delta \epsilon_{\text{PSF},i}|^2$. In this paper, with the introduction of biases, the scatter becomes the MSE:

$$ \text{MSE}[R_{\text{PSF}}^2] \simeq \left\langle |\delta R_{\text{PSF}}|^2 \right\rangle, $$

$$ \text{MSE}[\epsilon_{\text{PSF},i}] \simeq \left\langle |\delta \epsilon_{\text{PSF},i}|^2 \right\rangle. \quad (15) $$

This leads to:

$$ \sigma_{\text{sys}}^2 \simeq C \left[ b^2[\epsilon_{\text{PSF}},1] + \sigma^2[\epsilon_{\text{PSF}},1] + b^2[\epsilon_{\text{PSF}},2] + \sigma^2[\epsilon_{\text{PSF}},2] \right. $$

$$ + \left. \frac{\mathcal{E}}{R_{\text{PSF}}^2} \left( b^2[R_{\text{PSF}}^2] + \sigma^2[R_{\text{PSF}}^2] \right) \right]. \quad (16) $$

We can see that $\sigma_{\text{sys}}^2$ is proportional to the quadratic sum of 6 terms: three bias terms and three statistical ones. Collecting terms of the same kind using the following notation:

$$ B \equiv b^2[\epsilon_{\text{PSF}},1] + b^2[\epsilon_{\text{PSF}},2] + \mathcal{E} \frac{b^2[R_{\text{PSF}}^2]}{R_{\text{PSF}}^2}, \quad (17) $$

$$ \Sigma \equiv \sigma^2[\epsilon_{\text{PSF}},1] + \sigma^2[\epsilon_{\text{PSF}},2] + \mathcal{E} \frac{\sigma^2[R_{\text{PSF}}^2]}{R_{\text{PSF}}^2}, \quad (18) $$

gives:

$$ \sigma_{\text{sys}}^2 \simeq C \left[ B + \Sigma \right]. \quad (19) $$

B and $\Sigma$ both depend on the complexity of the modelling, but only $\Sigma$ depends on the Signal-to-Noise Ratio of the stars. They can be noted $B(\Psi)$ and $\Sigma(\Psi, S_{\text{eff}})$. In P1 we show that it is given by:

$$ \Sigma(\Psi, S_{\text{eff}}) = \frac{q^2}{N_s S_{\text{eff}}}, \quad (20) $$

where the overall complexity of the model $\Psi$ is given by the complexities $q$, $\psi$, and $\psi_{R^2}$ associated to the ellipticity and the squared radius of the model respectively (see equation [2]):

$$ \Psi^2 = 2q^2 + \mathcal{E} \psi_{R^2}. \quad (21) $$
Equations 19 and 20 then give:

\[
\sigma_{sys}^2 \simeq C \left( B(\Psi) + \frac{\Psi^2}{N_s S_{eff}} \right). \tag{22}
\]

From this equation, we see that increasing the complexity \( \Psi \) by adding new degrees of freedom in the PSF model can reduce \( B \) but also increases the statistical errors. Minimising \( \sigma_{sys}^2 \) thus implies the search for the optimal trade-off for the value of \( \Psi \).

4. Optimal PSF model

In section 4.1 we present a PSF example that we use for the rest of this paper to illustrate our discussion. We show in section 4.2 how to derive the optimal complexity of the PSF model (that minimises \( \sigma_{sys}^2 \)) and apply this to the PSF example. We then explore this optimisation in more detail in section 4.3 by examining the particular case where the bias can be described as a power law of the complexity.

4.1. PSF example

In order to illustrate the discussion, we study a realistic PSF example with complex features in the tails and investigate what happens when fitting it with various shapelet basis sets as function of the SNR of the available stars. We also use a shapelet basis set (different from that used for the fits) for describing the example of underlying PSF. This use of shapelets both for the PSF model and the underlying PSF was chosen for three reasons:

- first, it allows one to ignore pixelation issues, which are beyond the scope of this paper. Indeed, the description of the underlying PSF is performed through the continuous shapelet functions and the fits are performed in very high resolution;
- second, it considerably simplifies the calculations and the fitting process, due to the orthogonality of shapelet functions (the average estimation of a fitted coefficient is the true value, independently of the other coefficients);
- third, it is a simple and convenient framework to illustrate the use of the sparsity as a tool for optimising the complexity of the PSF modelling.

Our example of underlying PSF is constructed using \( n_{max} = 34 \) (with the diamond option), as shown in figure 1. The 16 fits performed with the 4 shapelet basis sets corresponding to \( n_{max} = 4, 6, 10, 20 \) and for \( \sqrt{n_s S_{eff}} \) (the total signal-to-noise available from stars, see equation 3) equal to 100, 10^3, 10^4 or \( \infty \) (the latter is the ideal case of no background), are also shown in figure 1. In order to fix the overall complexity \( \Psi \) which depends on the rms of galaxy ellipticities through the parameter \( E \) (see equations 13 and 21) we adopt the typical value \( E = 0.2 \), for which \( n_{max} = 4, 6, 10, 20, 34 \) correspond to \( \Psi = 2.6, 4.3, 7.8, 16.4, 28.4 \) respectively. In the following we also adopt the value \( C = 0.066 \) that corresponds to the typical values \( P_\gamma = 1.84 \) and \( \left[ \left( \frac{R_{eff}}{R_{sys}} \right)^4 \right]^{1/4} = 1.5 \).

Figure 1 illustrates that:

- as long as \( \sqrt{n_s S_{eff}} \) is large enough, complex basis sets are required to model the complex tails, i.e. the amount of bias \( B \) decreases when the complexity \( \Psi \) of the model increases. For instance, a fit with \( S_{eff} = \infty \) and \( \Psi = 28.4 \) would allow one to recover exactly our PSF example with \( B = 0 \).
- a higher complexity means a larger number of DoF to be fitted. Consequently, for a given value of \( S_{eff} \), increasing the complexity of the model also increases the scatter of the estimated shape. Therefore, it is not necessarily interesting to use a complex fit model. It may be more robust to use a simplified (but more biased) fit model.

4.2. Optimising the complexity of the PSF model

The optimal PSF model is that for which \( \sigma_{sys} \) is minimal when varying \( \Psi \). We define the optimal value \( \sigma_{sys}^{opt} \) as the minimal value of \( \sigma_{sys} \) and in the same spirit, we note \( \Psi_{opt} \) the corresponding value of \( \Psi \):

\[
\sigma_{sys}^{opt} \equiv \sigma_{sys} \text{ such that } \left. \frac{\partial (\sigma_{sys}^2)}{\partial \Psi} \right|_{\Psi_{opt}} = 0. \tag{23}
\]

For instance, figure 2 illustrates the search for the optimal shapelet basis when our PSF example (see the previous section) is estimated with 50 stars (\( S_{eff} = 1000 \)). The optimal model is that corresponding to \( \Psi_{opt} \simeq 6 \) (i.e. \( n_{max} = 8 \) with the diamond option) and \( \sigma_{sys}^{opt} \simeq 10^{-7} \), shown by the red diamond.

For a given fit model, increasing \( N_s \) reduces the scatter but not the biases of the model fitting (see equation 22). Therefore, when \( N_s \) increases, \( \Psi_{opt} \) increases and \( \sigma_{sys}^{opt} \) decreases. This is illustrated in figure 3 which shows \( B, \Sigma \) and \( \sigma_{sys} \) (see equations 17 to 19) when our PSF example is estimated with \( N_s = 10, 50 \) and 200 (still \( S_{eff} = 1000 \)).

Figure 4 shows \( \sigma_{sys}^{opt} \) as a function of \( N_s \). The diamonds present the curve for our PSF example illustrated in all previous plots, while the straight bold line without diamonds present the curve for our PSF example illustrated in the previous section.

4.3. Example of the use of sparsity in the case of a power law

In this section we derive the optimal complexity when the bias \( B \) is a power law of the complexity written as:
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Fig. 1. PSF example (top panel) adopted in this paper and best fits of it (other panels) with 4 shapelet basis sets (corresponding to $\Psi_{\text{fit}}$ = 2.6, 4.3, 7.8 and 16.4, i.e. $n_{\text{max}}$ = 4, 6, 10 and 20 with the diamond option) and for $\sqrt{n_s S_{\text{eff}}}$ equal to 100, $10^3$, $10^4$ and $\infty$ ($n_s$ is the numbers of stars used for the fit and $S_{\text{eff}}$ is the effective SNR of stars, see equation\[K\] infinity corresponds to the ideal case of no background). For a given $S_{\text{eff}}$, all fits are performed with the same realisation of the noise. Colors show the flux (darker colors indicate brighter regions) and show that this PSF is almost circular at the center. Contours show some isophotes not visible with the color scale and reveal the complex structure of the tails. The original (i.e. underlying) PSF in the top panel was built using a model with $\Psi$ = 24.8. The optimal value $\Psi_{\text{opt}}$ of the fitted complexity (in order to minimise $\sigma_{\text{sys}}$) is indicated under brackets for each value of $S_{\text{eff}}$. This figure illustrates that, for a given $S_{\text{eff}}$, the simpler the model (i.e. the lower $\Psi_{\text{fit}}$), the poorer the description of the tails and the larger the bias. On the other hand, for a given $\Psi_{\text{fit}}$, the lower the Signal-to-Noise (i.e. the smaller $S_{\text{eff}}$), the lower the amount of information available for the fit and the noisier the description of the tails.
PSF without large residuals. In the following, a large number of parameters to describe the underlying representation (i.e. equation 24), equations 22 and 23 imply the ‘sparsity parameter’. Together with this power law parameters. Conversely a small value of Ψ that for which σ_{sys} is minimum. At this point, shown by the red diamond, σ_{sys} ≡ σ^{opt}_{sys} ≃ 10^{-7} and Ψ ≡ Ψ^{opt} ≃ 6.

\[ B \propto 1/\Psi^{\alpha}. \]

We investigate \((σ^{opt}_{sys})^2\) as a function of \(N_∗\) and \(α\). We normalise the power law such as:

\[ B(Ψ) = B_0 \left( \frac{Ψ_0}{Ψ} \right)^{\alpha}. \]  

In our example of a PSF fitted with a shapelet basis set (see section 4.1 and figure 1), the smallest value of Ψ that we probe is 2.6 (this corresponds to \(n_{\text{max}} = 4\) with the ‘diamond’ configuration, see P1). That is why, for this example, we choose to normalise the power law at this value Ψ_0 = 2.6, implying \(B_0 = 2 \times 10^{-5}\). This model is illustrated in figure 2 for \(α\) equal to 2, 4, 6 and 8. This is superimposed on the \(B\) versus Ψ relation we obtain when fitting our PSF example with different shapelet basis sets as described in section 4.1. We see that in this case \(B\) is reasonably described by \(α = 4\).

This representation with a power law is particularly convenient because \(α\) can be identified with the sparsity: a large \(α\) means the PSF model is efficient to represent the underlying PSF with a small number of free parameters. Conversely a small \(α\) means the PSF model needs a large number of parameters to describe the underlying PSF without large residuals. In the following, \(α\) is called the ‘sparsity parameter’. Together with this power law representation (i.e. equation 24), equations 22 and 23 imply:

\[ (σ^{opt}_{sys}(Ψ))^2 = C \left[ B_0 \left( \frac{Ψ_0}{Ψ} \right)^{α} + \frac{Ψ^2}{N_∗ S^2_{\text{eff}}} \right], \]

\[ Ψ^{opt} = \left[ α B_0 Ψ_0 ^2 N_∗ S^2_{\text{eff}}/2 \right]^{1/(α+2)}, \]

\[ (σ^{opt}_{sys})^2 = (σ_{sys}(Ψ = Ψ^{opt}))^2 \]

\[ = C \left[ B_0^2 \left( \frac{2 Ψ_0^2}{α N_∗ S^2_{\text{eff}}} \right)^{α} \right]^{1/(α+2)}. \]

Note that equation 27 gives \((σ^{opt}_{sys})^2\) (the minimum variance of the systematic errors in shear measurements that can be reached, see equations 19, 20 and 23) in terms of a set of parameters that can be divided into 2 families:

1. parameters that are properties of the data set, which are \(C, B_0, Ψ_0\) and \(S_{\text{eff}}\).

2. parameters that are properties of the analysis method, which are \(N_∗\) (i.e. the number of stars used to calibrate the PSF) and \(α\) (i.e. the sparsity parameter of the PSF model).

When analysing a given data set, parameters of the first family are fixed. The only parameters that can be opti-
Fig. 4. Optimal variance of the shape measurement systematics \((\sigma_{\text{sys}}^{\text{opt}})^2\) as a function of the number of stars \(N_\star\). The diamonds show the curve for our PSF example illustrated in all previous plots (presented on figure 1) and fitted with shapelets. The straight bold line shows the ideal case (addressed in P1) of a PSF model that exactly describes the PSF (i.e. with no residual). The horizontal line shows the values \((\sigma_{\text{sys}}^{\text{opt}})^2 = 10^{-7}\) which is the requirement to be able to constraint \(w_0\) and \(w_a\) at 0.02 and 0.1 respectively (Amara & Refregier 2007b). The blue lines (dashed, dotted and dotted-dashed) are discussed in section 4.3. They are the curves expected when modelling the bias \(B\) with a power law of the complexity as stated by equation 24 and illustrated in figure 5.

\[
\sigma_{\text{sys}}^{\text{opt}} \sim 50 \left( \frac{S_{\text{eff}}}{500} \right)^{-2} \left( \frac{R_{\text{gal}}}{R_{\text{PSF}}^{\text{min}}} \right)_{1.5}^{-4} \left( \frac{\sigma_{\text{sys}}^{2}}{10^{-7}} \right)^{-1} \left( \frac{\Psi}{3} \right)^2/2. \tag{28}
\]

The factor 2 at the end comes from the fact that \(\Psi^2 \simeq 2\psi_c^2\) (in P1 this scaling relation is written in terms of \(\psi_c\)). This holds with the assumption that the PSF model is able to describe the PSF without any bias (i.e. \(B = 0\)). With non-zero \(B\) and adopting the same simplifications and the same typical values than in P1, equation 22 leads to the more general relation:

\[
N_\star \simeq 50 \left( \frac{S_{\text{eff}}}{500} \right)^{-2} \left( \frac{R_{\text{gal}}}{R_{\text{PSF}}^{\text{min}}} \right)_{1.5}^{-4} \left( \frac{\sigma_{\text{sys}}^{2}}{10^{-7}} \right)^{-1} \Phi^2, \tag{29}
\]

5. Required number of stars

As discussed in the introduction, one issue for cosmic shear surveys is to ensure that systematics are below statistical errors by imposing an upper limit on \(\sigma_{\text{sys}}\). Part of the systematics come from the PSF calibration, which is imperfect due to the limited number of stars available. In this section we express \(N_\star\), the number of stars required to calibrate the PSF, in terms of the level of systematics \(\sigma_{\text{sys}}\). In section 5.1 we summarise the conclusions of P1 that hold when the underlying PSF and the PSF model have the same functional form (i.e. \(B = 0\)) and we extend these conclusions to the general case of PSF modelling performed with any model (i.e. \(B\) not necessarily equal to 0). In section 5.2 we invert equation 27 (that holds for the case of \(B\) described by a power law of the complexity: \(B \propto 1/\Psi^\alpha\)) and express \(N_\star\) as a function of \(\alpha\) and of the minimum systematic level \(\sigma_{\text{sys}}^{\text{opt}}\) achievable when the complexity of the PSF modelling is optimal.

5.1. Generalised scaling relation

In the optimistic case where the PSF calibration is the only significant source of systematics, a given value of \(N_\star\) (i.e. a given number of stars involved in the PSF calibration) implies a value of \(\sigma_{\text{sys}}\). This is presented in P1 in the form of a scaling relation that links \(N_\star\), \(\sigma_{\text{sys}}\), \(S_{\text{eff}}\) (the effective Signal-to-Noise Ratio of stars), \((R_{\text{gal}}/R_{\text{PSF}})^{\text{min}}\) (the ratio between the smallest galaxy size and the PSF size) and \(\Psi\) (the complexity of the PSF):

\[
N_\star \simeq 50 \left( \frac{S_{\text{eff}}}{500} \right)^{-2} \left( \frac{R_{\text{gal}}}{R_{\text{PSF}}^{\text{min}}} \right)_{1.5}^{-4} \left( \frac{\sigma_{\text{sys}}^{2}}{10^{-7}} \right)^{-1} \left( \frac{\Psi}{3} \right)^2/2. \tag{28}
\]
Thus, taking $B$ into account in the scaling relation translates into the new factor $1/ \left[ 1 - C \frac{B}{\sigma_{\text{sys}}} \right]$ in equation (30), which is equal to 1 when $B$ is null (then the relation 29 is equivalent to the scaling relation given in P1) and is related to the ratio $\frac{B}{\sigma_{\text{sys}}}$, which is the relative weight of biases in the error budget.

5.2. Application to the power law model

Equation 27 can be inverted to give $N_*$ (the number of stars required to calibrate the PSF) as a function of $\sigma_{\text{sys}}^{\text{opt}}$ (the minimum level of systematics achievable when optimising the complexity of the PSF modelling), $\alpha$ (the sparsity parameter), $S_{\text{eff}}$ (the effective Signal-to-Noise Ratio of stars defined in equation 3) and $C$ (a dimensionless factor defined in equation 12):

$$N_* = \frac{\Psi_0^2 B_0^{2/\alpha}}{S_{\text{eff}}^2} \left( \frac{C}{\sigma_{\text{sys}}^{\text{opt}}} \right)^{1+2/\alpha} h(\alpha) \left( \frac{\sigma_{\text{sys}}^{\text{opt}}}{\sigma_{\text{sys}}} \right)^{2(1+2/\alpha)}$$

with the dimensionless function:

$$h(\alpha) = \left( \frac{\alpha}{2} \right)^{2/\alpha} \left( 1 + \frac{2}{\alpha} \right)^{1+2/\alpha}$$

shown in figure 6. With the notation and scaling of equation 29, equation 31 is equivalent to:

$$N_* = 16 \left( \frac{\Psi_0}{2.6} \right)^2 \left( 12 \frac{B_0}{2 \times 10^{-5}} \right)^{2/\alpha} h(\alpha) \left( \frac{S_{\text{eff}}}{500} \right)^{-2} \times \left( \frac{R_{\text{gal}}/R_{\text{PSF}}}{1.5} \right)^{-4(1+2/\alpha)} \left( \frac{\sigma_{\text{sys}}^{\text{opt}}}{10^{-7}} \right)^{-(1+2/\alpha)}$$

This equation allows one to estimate the number of stars required to calibrate the PSF and thus, with respect to the stellar density, the minimum scale for which the PSF calibration is possible. On smaller scales, stars do not provide enough information to calibrate PSF variations. That means these smaller scales may be contaminated by systematics due to a poor correction of the PSF and should not be used to estimate cosmological parameters, unless the variabilities at small scales are known to be extremely small. As shown by Amara & Refregier (2007b) and discussed in P1, future all-sky cosmic shear surveys will need to achieve $(\sigma_{\text{sys}}^{\text{opt}})^2 \leq 10^{-7}$ in order to get estimations of $w_0$ and $w_a$ with error bars of about 0.02 and 0.1 respectively. Figure 4 shows that, for our PSF example, it is possible to achieve this requirement when calibrating the PSF with 50 stars, if $\alpha \geq 4$. Although this is not a general statement (this assumes $B$ can be described by a power law, see equation 24 and depends on the PSF through $B_0$), this is a representative example of the sparsity requirement for future cosmic shear surveys. On the other hand, for current cosmic shear surveys of $\sim 50$ deg$^2$, we have the requirement $(\sigma_{\text{sys}}^{\text{opt}})^2 \leq 4 \times 10^{-6}$ (see Amara & Refregier (2007b) and P1). In this case, it is possible to calibrate the PSF with a couple of stars when $\alpha \geq 2$. This sparsity requirement is reached with the current PSF correction methods (for instance that based on shapelets as in Bergé et al. 2008 and, assuming a star density of about 1 per arcmin$^2$, this is compatible with significant $B$ modes usually found at scales smaller than few arcmins.

6. Conclusions

We explore the systematics induced in cosmic shear by the PSF calibration-correction process and study how to optimise the PSF model in order to minimise the systematics on cosmological parameter estimations. In this framework, we revisite the concept of complexity of the PSF model, defined in a previous paper (P1) and introduce the concept of ‘sparsity’ of the PSF model. The complexity $\Psi$ characterises the number of degrees of freedom in the model. A small number of degrees of freedom corresponds to a low $\Psi$ and relates to a simple PSF model which can be fitted on low Signal-to-Noise stars but is likely to be highly biased. On the opposite, a large number of degrees of freedom corresponds to a high $\Psi$ and relates to a complex PSF model which is expected to have a low bias but requires high Signal-to-Noise stars in order to avoid large statistical scatters of the fitted parameters. In P1, we relate the complexity $\Psi$ of the PSF model to the systematics on cosmological parameter estimations. In this paper, we...
show how to optimise the complexity according to the stars available, through the concept of sparsity. The sparsity characterises the decrease of residuals between the best fit of the PSF model and the underlying PSF, when adding new degrees of freedom in the model.

We also extend in the general case the scaling relation, proposed in P1, between the number of stars used to calibrate the PSF and the systematics on cosmological parameter estimations. As discussed in P1, this relation, with the constraint of maintaining the systematics under the statistical uncertainties when estimating cosmological parameters, gives the required number of stars \( N_\ast \) for the PSF calibration. \( N_\ast \) indicates the minimum scale the PSF modelling can access: on scales smaller than the minimum scale encompassing \( N_\ast \) stars, there is not enough information in the data to calibrate PSF variations. That means these smaller scales may be contaminated by systematics due to a poor PSF correction and should not be used when estimating cosmological parameters (unless the variabilities are known to be very small, thanks to the hardware).

We consider a realistic PSF example and model the amount of bias \( B \) between the PSF fit and the underlying PSF with a power law of the fitted complexity: \( B \propto 1/\Psi^\alpha \), where \( \alpha \) is the sparsity parameter. We find that, for this PSF, current cosmic shear analyses that cover 50 deg\(^2\) or less need \( \alpha \) to be greater than 2, which is reasonably reached by current analysis methods. Thus, current cosmic shear analyses do not require a hard optimisation of the PSF model. On the other hand, future cosmic shear surveys that aim to measure \( w_0 \) and \( w_a \) with an accuracy of 0.02 and 0.1 respectively require \( \alpha \geq 4 \) in order to calibrate the PSF with 50 stars. This relation between the required number of stars \( N_\ast \) and the accuracy of the calibration depends on the underlying PSF. That is why these values, although they provide realistic orders of magnitude, can not be considered as a general result. Two parameters drive this relation: the amount of biases \( B_0 \) when fitting the underlying PSF with a PSF model which has a low complexity (in our example, \( B_0 = 2 \times 10^{-5} \)), the required number of stars \( N_\ast \) is proportional to \( B_0^{2/\alpha} \); and the sparsity parameter \( \alpha \) of the PSF modelling during the analysis. It is thus possible to optimise cosmic shear surveys at two levels: when optimising the observational conditions, the PSF must be as simple and stable as possible in order to make possible its description by a low complexity model (this minimises \( B_0 \)); when analysing the data, the PSF modelling must be optimised in order to have as high a sparsity \( \alpha \) as possible.

The approach suggested in this paper is a first step for introducing the concept of sparsity for weak lensing shape measurements. We do not address the issues linked to the pixelation. Moreover, although we only address here the PSF calibration, this approach is also applicable to other usages such as description of galaxy shapes.

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