Hi John,

In case you're interested, I started with two normally-distributed random variables \( X, Y \sim N(0,1) \) then applied the following transformation to generate new random variables \( U, V \):

\[
U = \{ |X+Y|, -|X+Y| \} \\
V = \{ X+Y, -(X+Y) \}
\]

The notation here means that the vector formed by \(|X+Y|\) is joined with \(-|X+Y|\) to make a longer vector (\(= U\)) etc..

This transformation leads to marginalized normal distributions in each of \( U \) and \( V \) (also if projected onto each axis) and \( \text{cov}(U,V) = 0 \), i.e., they are uncorrelated. However \( U \) and \( V \) are dependent (see figure below).

Therefore, this is any example where two variables \((U,V)\) are not joint-normal, have covariance \(= 0\) but are indeed dependent. It reinforces the fact that the covariance measure completely determines (in)dependency between variables if and only if they are joint-normally distributed.

As a side note, joint-normally distributed variables can only be linearly-dependent. That's because ellipses and ellipsoids are the 'norm' for multinormal distributions. This is what the covariance can ever measure for joint-normally distributed data. E.g., Pearson's product moment coefficient is sufficient for computing the correlation between normally distributed variables because it measures the degree of linear-dependence only. For a more general measure of dependency between variables (e.g., related to high polynomial order), Spearman's rank correlation coefficient is the best choice.

Regards, Frank