

LETTER TO THE EDITOR

On two-dimensional fractional Brownian motion and fractional Brownian random field

Hong Qian^{†‡}, Gary M Raymond[‡] and James B Bassingthwaite[‡][†] Department of Applied Mathematics, University of Washington, Seattle, WA 98195, USA[‡] Department of Bioengineering, University of Washington, Seattle, WA 98195, USA

Received 25 March 1998, in final form 1 June 1998

Abstract. As a generalization of one-dimensional fractional Brownian motion (1dfBm), we introduce a class of two-dimensional, self-similar, strongly correlated random walks whose variance scales with power law N^{2H} ($0 < H < 1$). We report analytical results on the statistical size and shape, and segment distribution of its trajectory in the limit of large N . The relevance of these results to polymer theory is discussed. We also study the basic properties of a second generalization of 1dfBm, the two-dimensional fractional Brownian random field (2dfBrf). It is shown that the product of two 1dfBms is the only 2dfBrf which satisfies the self-similarity defined by Sinai.

1. Introduction

Random walk and Brownian motion have been ubiquitous models in physical and biological sciences (Wax 1954, Berg 1983). The theory of Brownian motion either focuses on the nonstationary Gaussian process or treats it as the sum of a stationary pure random process—the white noise in Langevin’s approach (Fox 1978). Generalization of the classical theory to the fractional Brownian motion (fBm) (Kahane 1985, Beran 1994), and the corresponding fractional Gaussian noise (fGn), now includes a wide class of self-similar stochastic processes whose variances scale with power law N^{2H} , where N is the number of steps in the fBm and $0 < H < 1$; $H = 0.5$ corresponds to the classical Brownian motion with independent steps. The term ‘fractional’ was proposed by Mandelbrot (Mandelbrot and van Ness 1968) in connection with fractional integration and differentiation. In general the steps in the fBm are strongly correlated and have long memory. Hence, fBm has become a powerful mathematical model for studying correlated random motion with wide application in physics and biology (Ding and Yang 1995, Vlad *et al* 1996). In this letter, we generalize the one-dimensional fBm to two dimensions (2D). We consider both a stochastic process in 2D, i.e. a sequence of two-dimensional random vectors $\{(B_i^x, B_i^y) | i = 0, 1, \dots\}$ and a random field in 2D $\{B_{i,j} | i = 0, 1, \dots; j = 0, 1, \dots\}$ which is a scalar random variable defined on a plane. They correspond to the spin dimension 2 and space dimension 2, respectively, in the theory of critical phenomena in statistical physics (Ma 1976). The most general fractional Brownian random field will be a n -dimensional random vector defined in a m -dimensional space (Kahane 1985), and there have been studies on such a model in pure mathematics (e.g. Xiao 1997).

2. The size and shape of two-dimensional fBm

In two-dimensional fBm (2dfBm), B^x and B^y are two identical but independent one-dimensional fBms (1dfBms). The defining property of a 1dfBm, B_k , is its autocorrelation function (Beran 1994):

$$E[B_k B_h] = \langle B_k B_h \rangle = \frac{\sigma^2}{2} [h^{2H} - (h-k)^{2H} + k^{2H}].$$

Here $E[\cdot]$ and $\langle \cdot \rangle$ are notations for expectation value in mathematics and ensemble average in physics, respectively. Now let's consider a N -step trajectory of the 2dfBm:

$$(u_0, v_0) = (0, 0), (u_1, v_1), (u_2, v_2), \dots, (u_k, v_k), \dots, (u_N, v_N). \quad (1)$$

We are interested in the geometry of this trajectory, i.e. the 'shape' of the N points in (1), in a statistical sense. We define the random moment matrix (radius of gyration tensor):

$$\begin{pmatrix} \theta_{uu} & \theta_{uv} \\ \theta_{vu} & \theta_{vv} \end{pmatrix} \quad (2)$$

where

$$\begin{aligned} \theta_{uu} &= \frac{\sum_{k=1}^N u_k^2}{N} - \left(\frac{\sum_{k=1}^N u_k}{N} \right)^2 & \theta_{vv} &= \frac{\sum_{k=1}^N v_k^2}{N} - \left(\frac{\sum_{k=1}^N v_k}{N} \right)^2 \\ \theta_{uv} = \theta_{vu} &= \frac{\sum_{k=1}^N u_k v_k}{N} - \frac{\sum_{k=1}^N u_k \sum_{h=1}^N v_h}{N^2}. \end{aligned}$$

The eigenvalues of the matrix, λ_1 and λ_2 , are known as the squares of the principal components of the radius of gyration of the N -point object:

$$\lambda_1 + \lambda_2 = \theta_{uu} + \theta_{vv} \quad (3)$$

$$|\lambda_1 - \lambda_2| = \sqrt{(\theta_{uu} - \theta_{vv})^2 + 4\theta_{uv}^2}. \quad (4)$$

The eigenvalues characterize the size of the object, whose expectation (ensemble average) is:

$$\left\langle \frac{\lambda_1 + \lambda_2}{2} \right\rangle = \frac{\sigma^2}{2N^2} \sum_{h=1}^N \sum_{k=1}^N (h-k)^{2H} \sim \frac{\sigma^2 N^{2H}}{(2H+1)(2H+2)} \quad (N \rightarrow +\infty) \quad (5)$$

where $\sigma^2 = \text{Var}[B_x] = \text{Var}[B_y]$. For $H = \frac{1}{2}$ the radius of gyration is $N\sigma^2/6$ which is well known in polymer theory (Flory 1969)†. The eigenvalues also characterize the asymmetry of the object:

$$\frac{|\lambda_1 - \lambda_2|}{\lambda_1 + \lambda_2}. \quad (6)$$

Unfortunately, the expectation of (6) is difficult to compute. However, it is relatively easy to compute the expectations of the square of the numerator and denominator separately, which leads to the asymmetry A_{sy} proposed by Rudnick and Gaspari (1987):

$$A_{sy} = \frac{\langle (\lambda_1 - \lambda_2)^2 \rangle}{\langle (\lambda_1 + \lambda_2)^2 \rangle}. \quad (7)$$

† When an excluded volume, i.e. steric hindrances, is introduced, the so-called self-avoiding walks (SAW) have $H = 3/(d+2)$ where d is the dimension of the Euclid space for the polymer (de Gennes 1979). It is obvious that the fBm is not a faithful model for the SAW in 2D. This raises an interesting question: since the fBm is shown by Sinai (1976) to be the unique stochastic process possessing self-similarity, what is the relation between fBm and SAW? This problem is intimately related to the mathematical problem of self-crossing and intersectional local time where the significance of $H = \frac{3}{4}$ has been noted (Rosen 1987).

Detailed calculation shows that in the limit of large N (appendix A):

$$A_{sy} \approx 2 - \frac{1}{\frac{2(H+1)^2}{\frac{1}{2(H+1)^2} + \frac{2H+1}{4(4H+1)} - \frac{1}{4H+3} - \frac{\Gamma^2(2H+2)}{\Gamma(4H+4)}}} \quad (8)$$

Therefore, $A_{sy} = \frac{4}{7}$ ($= 0.57$), 0.81 , and 1 for $H = \frac{1}{2}$, $\frac{3}{4}$, and 1 , respectively. $\frac{4}{7}$ is in agreement with the previous calculation $2(d+2)/(5d+4)$ for d -dimensional random walks when $d = 2$ (Rudnick and Gaspari 1987).

3. The distribution of the segments of 2dfBm

In the previous section, we studied the radius of gyration tensor of the finite trajectory of a 2dfBm, which characterize the size and shape of the trajectory. In this section, we study the distribution of the segments in a trajectory with respect to the centre of gravity (Debye and Bueche 1952). For this analysis, it is sufficient to obtain the result for 1dfBm $\{B_k | k = 0, 1, \dots, N-1\}$. Let us fix our coordinate system to the center of gravity of the N points:

$$\hat{B}_k = B_k - \frac{B_0 + B_1 + \dots + B_{N-1}}{N}$$

where the 1dfBm can be expressed in terms of the partial sum of fGn X_ℓ (Beran 1994):

$$B_k = X_1 + X_2 + X_3 + \dots + X_k \quad B_0 = 0$$

and without loss the generality, we assume that $\langle X_\ell \rangle = 0$ and denote $\sigma^2 = \langle X_\ell^2 \rangle$ for all $1 \leq \ell \leq N-1$. Hence,

$$N\hat{B}_k = X_1 + 2X_2 + \dots + (k-1)X_{k-1} - (N-k)X_k - [N-(k+1)]X_{k+1} \dots - [N-(N-1)]X_{N-1}.$$

It is convenient to introduce a matrix relation between X 's and \hat{B} 's (Yeh and Isihara 1969):

$$\hat{B}_k = \sum_{\ell=1}^{N-1} \psi_{k\ell} X_\ell \quad (9)$$

where

$$\psi_{k1} = 1/N, \psi_{k2} = 2/N, \dots, \psi_{k,k-1} = (k-1)/N$$

and

$$\begin{aligned} \psi_{k,k} &= -(N-k)/N, \psi_{k,k+1} = -[N-(k+1)]/N, \dots, \psi_{k,N-1} \\ &= -[N-(N-1)]/N = -1/N. \end{aligned}$$

Therefore, the random variable \hat{B}_k is Gaussian since it is the sum of $N-1$ Gaussian random variables, $\{X_\ell\}$, with a multivariate Gaussian distribution

$$P(\{X_\ell\}) = \frac{1}{(2\pi)^{(N-1)/2} D^{1/2}} \exp \left[-\frac{1}{2D} \sum_{i,j=1}^{N-1} \theta_{ij} X_i X_j \right] \quad (10)$$

in which θ_{ij} is the cofactor (inverse matrix) of the element (ij) in the matrix $[\gamma_{ij}]$ (Beran 1994):

$$\gamma_{ij} = \langle X_i X_j \rangle = \frac{\sigma^2}{2} [(i-j-1)^{2H} - 2(i-j)^{2H} + (i-j+1)^{2H}] \quad (11)$$

and D is the determinant of the matrix $[\gamma_{ij}]$. The following calculations for the mean and variance of \hat{B} 's only require the use of equation (11):

$$\langle \hat{B}_k \rangle = \left\langle \sum_{\ell=1}^{N-1} \psi_{k\ell} X_\ell \right\rangle = \sum_{\ell=1}^{N-1} \psi_{k\ell} \langle X_\ell \rangle = 0 \tag{12}$$

and

$$\begin{aligned} \langle \hat{B}_h \hat{B}_k \rangle &= \left\langle \sum_{\ell,m=1}^{N-1} \psi_{h\ell} \psi_{km} X_\ell X_m \right\rangle = \sum_{\ell,m=1}^{N-1} \psi_{h\ell} \psi_{km} \langle X_\ell X_m \rangle \\ &= \frac{\sigma^2}{2} \sum_{\ell,m=1}^{N-1} \psi_{h\ell} \psi_{km} [(\ell - m - 1)^{2H} - 2(\ell - m)^{2H} + (\ell - m + 1)^{2H}]. \end{aligned} \tag{13}$$

Therefore, the k th segment (step) in a N -step 2dfBm is Gaussian distributed with its expectation at the centre of gravity, and variance

$$\langle \hat{B}_k \rangle_{2d} = \sigma^2 \sum_{\ell,m=1}^{N-1} \psi_{k\ell} \psi_{km} [(\ell - m - 1)^{2H} - 2(\ell - m)^{2H} + (\ell - m + 1)^{2H}]. \tag{14}$$

Figure 1 shows the mean square distance between the k th segment and the centre of gravity for various 2dfBm with different H and N . This distance reaches the minimal at $k = N/2$. The figure also indicates that for large N , the distance is asymptotically proportional to N^{2H} . In fact, we have the analytical result:

$$\langle \hat{B}_k \rangle_{2d} \sim 2H(2H - 1)\sigma^2 N^{2H} \int_0^1 \int_0^1 \frac{\psi_\eta(x)\psi_\eta(y)}{(x - y)^{2-2H}} dx dy \tag{15}$$

where $\eta = k/N$, and

$$\psi_\eta(x) = \begin{cases} x & 0 \leq x \leq \eta \\ x - 1 & 1 \geq x > \eta. \end{cases} \tag{16}$$

The integration in equation (15) is carried out in appendix B which leads to

$$\begin{aligned} \langle \hat{B}_k \rangle_{2d} &\sim 4H(2H - 1)\sigma^2 N^{2H} \int_0^1 \int_0^x \frac{\psi_\eta(x)\psi_\eta(y)}{(x - y)^{2-2H}} dy dx \quad (N \rightarrow +\infty) \\ &= 2\sigma^2 N^{2H} \left[\frac{(1 - k/N)^{2H+1} + (k/N)^{2H+1} - 1}{2H + 1} + \frac{1}{2H + 2} \right]. \end{aligned} \tag{17}$$

For $H = \frac{1}{2}$, our calculation is in agreement with the previous result on random polymer (Yeh and Isihara 1969):

$$\begin{aligned} \langle \hat{B}_k \rangle_{2d}^{H=1/2} &= \frac{\sigma^2}{3N^2} [N(N + 1)(2N + 1) + 6Nk^2 - 6N(N + 1)k] \\ &\approx \frac{2N\sigma^2}{3} \left[1 + 3 \left(\frac{k}{N} \right)^2 - 3 \left(\frac{k}{N} \right) \right]. \end{aligned} \tag{18}$$

Figure 1 shows that for $N = 10$, the asymptotic form is already quite accurate.

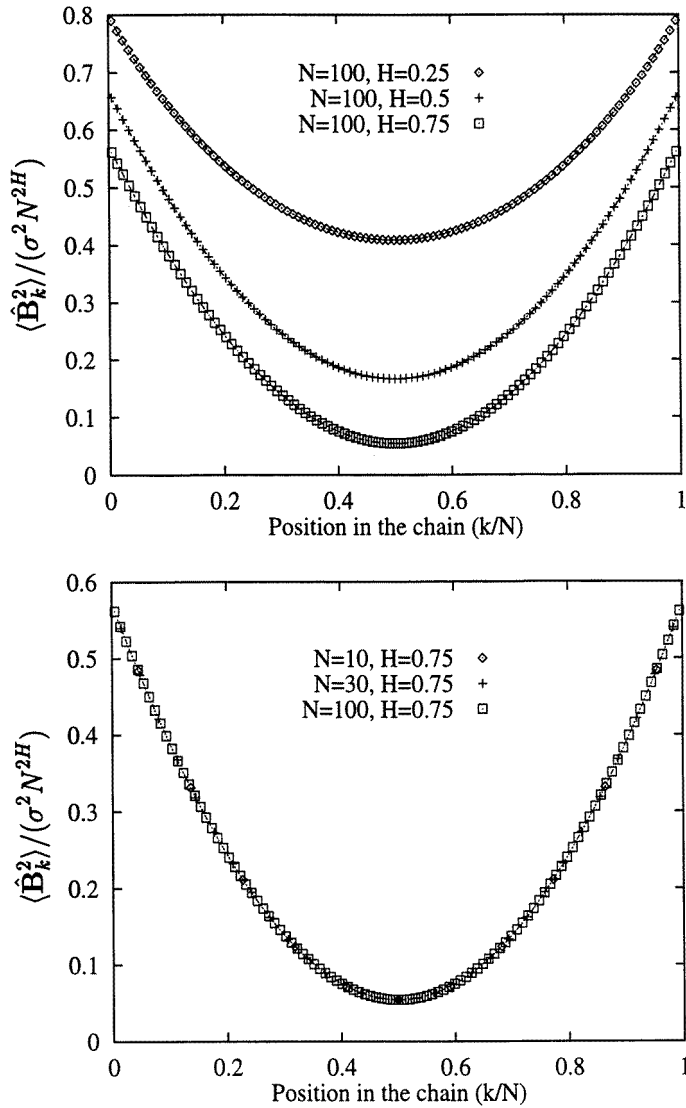


Figure 1. The mean square distance between the k th segment ($0 \leq k \leq N$) and the centre of gravity for a N -step 2dfBm with Hurst coefficient H . Various N and H are indicated in the figures. (a) Different H 's with same N according to equation (14). Completely superimposed with the three symbols are three additional curves: $2\{[(1-k/N)^{2H+1} + (k/N)^{2H+1} - 1]/(2H+1) + 1/(2H+2)\}$ (equation (17)) with $H = 0.25, 0.5,$ and $0.75,$ respectively. (b) Different N 's with same H . Completely superimposed with the symbols, there is also the curve $2\{[(1-k/N)^{2.5} + (k/N)^{2.5} - 1]/2.5 + \frac{1}{3.5}\}$, representing the asymptotics for large N .

4. The two-dimensional fractional Brownian random field

We now turn to a second generalization of 1dfBm. Following Sinai (1976), a self-similar, isotropic two-dimensional fGn is a matrix of identical Gaussian random variables $X_{1,1}, X_{1,2},$

$X_{2,1}, X_{2,2}, \dots$, which has the following defining property (Mandelbrot and van Ness 1968):

$$\frac{1}{(NM)^H} \sum_{i=1}^N \sum_{j=1}^M X_{i,j} \sim X_{1,1} \tag{19}$$

where ‘ \sim ’ denotes equality in the sense of probability distribution. It can be shown that when $H = \frac{1}{2}$, the X ’s are also independent. In general, $0 < H < 1$ and the X ’s are correlated.

The *two-dimensional fractional Brownian random field* (2dfBrf) is defined as the partial sum of the two-dimensional fGn:

$$B_{h,k} = \sum_{i=1}^h \sum_{j=1}^k X_{i,j}. \tag{20}$$

Hence, we have:

$$\langle (B_{h,m} - B_{h,k})^2 \rangle = h^{2H} (m - k)^{2H} \sigma^2 = (hm)^{2H} \sigma^2 + (hk)^{2H} \sigma^2 - 2 \langle B_{h,m} B_{h,k} \rangle$$

where we have assumed, without loss generality, $\langle X_{1,1} \rangle = 0$ and denote $\sigma^2 = \text{Var}[X_{1,1}] = \langle X_{1,1}^2 \rangle$. Therefore,

$$\begin{aligned} \langle B_{h,m} B_{h,k} \rangle &= \frac{\sigma^2}{2} h^{2H} [m^{2H} - (m - k)^{2H} + k^{2H}] \\ \langle B_{\ell,k} B_{h,k} \rangle &= \frac{\sigma^2}{2} [\ell^{2H} - (\ell - h)^{2H} + h^{2H}] k^{2H} \end{aligned}$$

and

$$\begin{aligned} \langle B_{\ell,m} B_{h,k} \rangle + \langle B_{\ell,k} B_{h,m} \rangle &= \frac{\sigma^2}{2} [(\ell m)^{2H} + (hk)^{2H} + (\ell k)^{2H} + (hm)^{2H} \\ &\quad - (m - k)^{2H} (\ell^{2H} + h^{2H}) - (\ell - h)^{2H} (m^{2H} + k^{2H}) + (\ell - h)^{2H} (m - k)^{2H}]. \end{aligned}$$

Noting that $\langle X_{hk} X_{\ell m} \rangle = \langle X_{hm} X_{\ell k} \rangle$ therefore $\langle B_{hk} B_{\ell m} \rangle = \langle B_{hm} B_{\ell k} \rangle$. Thus, we finally obtain the correlation for the 2dfBrf:

$$\begin{aligned} \langle B_{h,k} B_{\ell,m} \rangle &= \frac{\sigma^2}{4} [(\ell m)^{2H} + (hk)^{2H} + (\ell k)^{2H} + (hm)^{2H} - (m - k)^{2H} (\ell^{2H} + h^{2H}) \\ &\quad - (\ell - h)^{2H} (m^{2H} + k^{2H}) + (\ell - h)^{2H} (m - k)^{2H}] \\ &= \frac{\sigma^2}{4} [(\ell^{2H} - (\ell - h)^{2H} + h^{2H})][(m^{2H} - (m - k)^{2H} + k^{2H})]. \end{aligned} \tag{21}$$

It is interesting to point out that our $B_{h,k}$ is different from the random field studied by mathematicians (Kahane 1985).

One can also determine the correlation for the two-dimensional fGn:

$$\begin{aligned} \langle X_{h,k} X_{\ell,m} \rangle &= \langle (B_{h,k} - B_{h-1,k} - B_{h,k-1} + B_{h-1,k-1})(B_{\ell,m} - B_{\ell-1,m} - B_{\ell,m-1} + B_{\ell-1,m-1}) \rangle \\ &= \frac{\sigma^2}{4} [(\ell - h - 1)^{2H} - 2(\ell - h)^{2H} + (\ell - h + 1)^{2H}] \\ &\quad \times [(m - k - 1)^{2H} - 2(m - k)^{2H} + (m - k + 1)^{2H}]. \end{aligned} \tag{22}$$

Using the stationarity of X , we have:

$$\langle X_{0,0} X_{m,n} \rangle = \frac{\sigma^2}{4} [(m - 1)^{2H} - 2m^{2H} + (m + 1)^{2H}][(n - 1)^{2H} - 2n^{2H} + (n + 1)^{2H}] \tag{23}$$

$$= \sigma^2 \rho_H(m) \rho_H(n) \tag{24}$$

where $\rho_H(\cdot)$ is the normalized autocorrelation function for the one-dimensional fGn. Therefore, we have shown that the autocorrelation function for 2dfBm and two-dimensional fGn is simply the products of two 1dfBm and two one-dimensional fGn, respectively.

5. The spectral density of two-dimensional fractal Gaussian noise

Equation (23) leads to the power spectrum of a fGn:

$$\begin{aligned} S(f_1, f_2) &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \rho_H(m) \rho_H(n) e^{-2\pi i(f_1 n + f_2 m)} \\ &= S^H(f_1) S^H(f_2) \quad (f_1, f_2 \in [-\frac{1}{2}, \frac{1}{2}]) \end{aligned} \quad (25)$$

where $S^H(\cdot)$ is the spectral density for one-dimensional fGn, which has a singularity at $f = 0$ (Sinai 1976):

$$S^H(f) = C(1 - \cos(2\pi f)) \sum_{m=-\infty}^{\infty} \frac{1}{|f + m|^{2H+1}} \quad (26)$$

where C is a constant. Hence, the asymptotic behaviour of $S(f_1, f_2)$ at $f_1 = f_2 = 0$ is:

$$S(f_1, f_2) \sim C(f_1 f_2)^{1-2H}. \quad (27)$$

6. Discussion

Among the vast literatures on stochastic fractals (Bunde and Havlin 1994), few models match the mathematical elegance and analytical simplicity of the theory of fBm/fGn. As a natural generalization of the classical Brownian motion, therefore, fBm deserves further investigation as a model for strongly correlated physical processes. In this note, we have studied some basic geometrical properties of 2dfBm, and further generalization to three dimensions is conceptually straightforward, though algebraically more involved. It will be interesting to find out whether the three-dimensional theory is a useful model for polymers in good and bad solvents, where phenomenologically H tends to be greater or smaller than 0.5, respectively (Flory 1969, de Gennes 1979). The theoretical investigations on polymer chains have utilized a wide range of modern approaches, including field theory formalism (Edwards 1992), renormalization group (Aronovitz and Nelson 1986), series expansions (Nemirovsky *et al* 1992), and Monte Carlo simulation (Wittmer *et al* 1998).

We have also generalized the 1dfBm to a two-dimensional random field. Based on the self-similarity given by Sinai (1976), we have shown that the 2dfBrf is simply the product of two 1dfBms. Further generalization to higher dimensions is again straightforward. With these generalization, we expect that the mathematical theory of fBm will become an increasingly useful model in studying scientific problems ranging from the physics of membrane (Lipowski 1991) to correlated blood flow distribution in mammalian organs (Bassingthwaight *et al* 1994).

We would like to thank Professor Rich Bass, Professor Don Percival, Professor Ron Pyke, Professor Joe Rudnick and Professor David Thouless for discussions. This work was supported by the NCRR-NIH grant RR-1243.

Appendix A. Calculation of the asymmetry of 2dfBm

To compute equation (7), we first note some basic properties of Gaussian processes in general and fBm in particular. For a Gaussian stochastic process, all the joint distributions are multivariate Gaussian distributions. For any multivariate Gaussian random variables $\xi_1, \xi_2, \dots, \xi_N$ with $\langle \xi_k \rangle = 0$ ($k = 1, 2, \dots, N$), their moments with order higher than 2 can

all be expressed in terms of the second-order correlation (Cantrell 1970); this is known as the Wick's theorem in physics and Isserlis theorem in statistics (Reichl 1980):

$$\langle \xi_1 \xi_2 \dots \xi_N \rangle = \sum_P \prod_{j=1}^N \langle \xi_j, \xi_{P_j} \rangle \tag{28}$$

where the sum runs over *all* permutations $P: j \rightarrow P_j$ of the integers $j = 1, 2, \dots, N$. Specifically,

$$\langle \xi_h \xi_i \xi_j \xi_k \rangle = \langle \xi_h \xi_i \rangle \langle \xi_j \xi_k \rangle + \langle \xi_h \xi_j \rangle \langle \xi_i \xi_k \rangle + \langle \xi_h \xi_k \rangle \langle \xi_i \xi_j \rangle. \tag{29}$$

The second-order correlation for fBm with zero expectation is

$$\langle B_h B_k \rangle = \rho_H(h, k) = \frac{\sigma^2}{2} [h^{2H} - (h - k)^{2H} + k^{2H}]. \tag{30}$$

For $H = \frac{1}{2}$, we have $\rho_{1/2}(h, k) = \sigma^2 \min(h, k)$, and

$$\langle B_h B_k B_\ell B_m \rangle = \sigma^4 h(2k + \ell) \quad h \leq k \leq \ell \leq m. \tag{31}$$

Combining equations (7), (3), (4), (2), and (29), we have

$$\langle (\lambda_1 - \lambda_2)^2 \rangle = \frac{8}{N^2} \sum_{i,j} \rho_H^2(i, j) - \frac{16}{N^3} \sum_{i,j,k} \rho_H(i, k) \rho_H(j, k) + \frac{8}{N^4} \left(\sum_{i,j} \rho_H(i, j) \right)^2$$

and

$$\begin{aligned} \langle (\lambda_1 + \lambda_2)^2 \rangle &= \frac{4}{N^2} \left(\sum_i \rho_H(i, i) \right)^2 + \frac{4}{N^2} \sum_{i,j} \rho_H^2(i, j) - \frac{8}{N^3} \sum_{i,j} \rho_H(i, j) \sum_k \rho_H(k, k) \\ &\quad - \frac{8}{N^3} \sum_{i,j,k} \rho_H(i, k) \rho_H(j, k) + \frac{8}{N^4} \left(\sum_{i,j} \rho_H(i, j) \right)^2 \end{aligned}$$

in which

$$\rho_H(i, j) = \frac{\sigma^2}{2} [i^{2H} - (i - j)^{2H} + j^{2H}]. \tag{32}$$

Therefore using the Euler–Maclaurin summation formula we have large N asymptotics (Bender and Orszag 1978):

$$\begin{aligned} \sum_i \rho_H(i, i) &= \sigma^2 \sum_i i^{2H} \sim \frac{\sigma^2 N^{2H+1}}{2H+1} & \sum_{i,j} \rho_H(i, j) &\sim \frac{\sigma^2 N^{2H+2}}{2H+2} \\ \sum_{i,j} \rho_H^2(i, j) &\sim \frac{\sigma^4 N^{4H+2}}{2H+1} \left[\frac{4H+3}{4(4H+1)} - B(2H+1, 2H+2) \right] \\ \sum_{i,j,k} \rho_H(i, k) \rho_H(j, k) &\sim \frac{\sigma^4 N^{4H+3}}{2(2H+1)^2} \left[\frac{H}{H+1} + \frac{1}{4H+3} + \frac{(2H+1)^2}{2(4H+1)} \right. \\ &\quad \left. + B(2H+2, 2H+2) - (2H+1)B(2H+1, 2H+2) \right] \end{aligned}$$

where

$$B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

is the Beta function (Abramowitz and Stegun 1965). We finally have:

$$A_{sy} \approx 2 - \frac{\frac{1}{2(H+1)^2}}{\frac{1}{2(H+1)^2} + \frac{2H+1}{4(4H+1)} - \frac{1}{4H+3} - B(2H+2, 2H+2)}. \tag{33}$$

Appendix B. Calculation of the integration in equation (15)

Note that

$$\int_0^\eta \int_0^x \frac{xy \, dy \, dx}{(x-y)^{2\nu}} = \frac{\eta^{4-2\nu}}{(1-2\nu)(2-2\nu)(4-2\nu)}$$

and

$$\int_0^\eta x \, dx \int_\eta^1 \frac{(y-1) \, dy}{(x-y)^{2\nu}} = \frac{(1-\eta)^{3-2\nu} + \eta^{3-2\nu} - 1}{(1-2\nu)(2-2\nu)(3-2\nu)} - \frac{(1-\eta)^{4-2\nu} + \eta^{4-2\nu} - 1}{(1-2\nu)(2-2\nu)(4-2\nu)}.$$

Therefore,

$$\begin{aligned} \int_0^1 \int_0^x \frac{\psi_\eta(x)\psi_\eta(y)}{(x-y)^{2-2H}} \, dx \, dy &= \int_0^\eta \int_0^x \frac{xy \, dy \, dx}{(x-y)^{2-2H}} + \int_0^\eta \int_\eta^1 \frac{x(y-1) \, dy \, dx}{(x-y)^{2-2H}} \\ &\quad + \int_\eta^1 \int_\eta^x \frac{(x-1)(y-1) \, dy \, dx}{(x-y)^{2-2H}} \\ &= \int_0^\eta \int_0^x \frac{xy \, dy \, dx}{(x-y)^{2-2H}} + \int_0^\eta \int_\eta^1 \frac{x(y-1) \, dy \, dx}{(x-y)^{2-2H}} + \int_0^{1-\eta} \int_0^x \frac{xy \, dy \, dx}{(x-y)^{2-2H}} \\ &= \frac{(1-\eta)^{2H+1} + \eta^{2H+1} - 1}{(2H-1)(2H)(2H+1)} + \frac{1}{(2H-1)(2H)(2H+2)}. \end{aligned}$$

References

- Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)
- Aronovitz J A and Nelson D R 1986 *J. Physique* **47** 1445
- Bassingthwaighte J B, Liebovitch L S and West B J 1994 *Fractal Physiology* (London: Oxford University Press)
- Bender C M and Orszag S A 1978 *Advanced Mathematical Methods for Scientists and Engineers* (New York: McGraw-Hill)
- Beran J 1994 *Statistics for Long-memory Processes* (New York: Chapman and Hall)
- Berg H C 1983 *Random Walks in Biology* (Princeton, NJ: Princeton University Press)
- Bunde A and Havlin S 1994 *Fractals in Science* (Berlin: Springer)
- Cantrell C D 1970 *Phys. Rev. A* **1** 672
- Debye P and Bueche F 1952 *J. Chem. Phys.* **20** 1337
- de Gennes P G 1979 *Scaling Concepts in Polymer Physics* (Ithaca, NY: Cornell University Press)
- Ding M Z and Yang W M 1995 *Phys. Rev. E* **52** 207
- Edwards S F 1992 *Int. J. Mod. Phys. B* **6** 1579
- Flory P J 1969 *Statistical Mechanics of Chain Molecules* (New York: Interscience)
- Fox R F 1978 *Phys. Rep.* **48** 179
- Kahane J-P 1985 *Some Random Series of Functions* 2nd edn (London: Cambridge University Press)
- Lipowski R 1991 *Nature* **349** 475
- Ma S K 1976 *Modern Theory of Critical Phenomena* (Reading, MA: Benjamin)
- Mandelbrot B B and van Ness J W 1968 *SIAM Rev.* **10** 422
- Nemirovsky A M, Dudowicz J and Freed K F 1992 *J. Stat. Phys.* **67** 395
- Reichl L E 1980 *A Modern Course in Statistical Physics* (Austin, TX: University of Texas Press)
- Rosen J 1987 *J. Multivar. Anal.* **23** 37
- Rudnick J and Gaspari G 1987 *Science* **237** 384
- Sinai Ya G 1976 *Theor. Prob. Appl.* **21** 64
- Vlad M O, Schonfisch B and Mackey M C 1996 *Physica* **229A** 312
- Wax N 1954 *Selected Papers on Noise and Stochastic Processes* (New York: Dover)
- Wittmer J P, Milchev A and Cates M E 1998 *Europhys. Lett.* **41** 291
- Xiao Y M 1997 *Math. Proc. Camb. Phil. Soc.* **122** 565
- Yeh R and Ishihara A 1969 *J. Chem. Phys.* **51** 1215