Wide-field Infrared Survey Explorer (WISE)

PRex Subsystem Design Document

Version 1.0

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1 Introduction

1.1 Subsystem Overview

This document presents the requirements, design, algorithms, and state of implementation of the PRex (Position Reconstruction) subsystem of the WSDC data processing system. PRex consists of two major sections, SFPRex (Single-Frame Position Reconstruction) and MFPRex (Multi-Frame Position Reconstruction), which run in the single-scan and multi-scan pipelines, respectively.

1.1.1 Requirements

The PRex subsystem is required to match source detections corresponding to single sources on the sky observed in multiple frames, to use these groups of single-object multiple apparitions to compute frame geometry and sky position, to update FITS header data with the improved estimates, and to perform various bookkeeping functions. The Level 4 requirements supported by this processing are as follows.

> L4WSDC-014: The root mean square (1σ) error in WISE catalog positions with respect to 2MASS All-Sky Point Source Catalog positions shall be less than 0.5” on each axis, for sources with SNR > 20 in at least one WISE band.
>
> L4WSDC-017: The WISE Source Catalog shall contain equatorial (J2000) coordinates for objects detected in at least one band.
>
> L4WSDC-018: The WISE Source Catalog shall contain uncertainties in the coordinate measurements for each object.
>
> L4WSDC-046: The WSDS Pipeline processing shall reconstruct the J2000 equatorial positions of sources detected on the calibrated WISE images relative to the positions of objects in the 2MASS All-Sky Point Source Catalog that are detected in the WISE science images.
>
> L4WSDC-062: The WSDC shall perform quality analysis of all WISE science data and make reports available on a regular basis.
>
> L4WSDC-078: The WISE science data products shall use the International Celestial Reference System (ICRS) to describe the positions and motions of celestial bodies. WISE astrometry shall be mapped into the ICRS using the 2MASS All-Sky Point Source Catalog as the primary astrometric reference.

The SFPRex module is called by the WSDS Scan/Frame Pipeline and operates on a single frameset (i.e., a group of four single-band frames, one per wavelength channel, observed at the same sky position at the same time). Based on a discrepancy analysis of the sources observed and matched in the four channels (or fewer channels when some data are missing) and the 2MASS astrometric reference data, chi-square minimization solutions are obtained for five parameters in each channel: RA, Dec, Twist, SFX (scale factor in the X direction, across columns), and SFY (scale factor in the Y direction, across rows). Optical distortion is taken into account in mapping source positions from frame coordinates to sky coordinates.

The MFPRex module is called by the WSDS Multi-Frame Pipeline and operates on multiple framesets corresponding to overlapping coverages of a specified area on the sky. It also uses matched sets of sources and the 2MASS astrometric reference data to refine the registration of the frames relative to each other. It employs an interative method that minimizes local coordinate-projection distortion by treating overlapping framesets as nodes in a linked list while simultaneously considering differences with respect to 2MASS reference star positions. Discrepancy analysis of the matched sources is used to derive corrections to each frame's RA, Dec, and Twist.

1.1.2 Liens

- The 20-parameter fit, including uncertainty modeling, is not yet implemented in SFPRex.
- Outlier detection and rejection algorithms have not yet been designed.
- SIS's have not yet been written for any output files.
- MFPRex has not yet been coded.
- It has not been determined how to handle 2MASS position errors due to accumulated proper motion.

1.2 Applicable Documents

This subsystem conforms to the specifications in the following project documents:

- WISE Science Data Center Functional Requirements Document, WSDC D-R001
- WISE Science Data System Functional Design, WSDC D-D001
- Software Management Plan, WSDC D-M003

1.3 Acronyms

2MASS	Two-Micron All-Sky Survey
FOV	Field of View
FRD	Functional Requirements Document
MFPRex	Multi-Frame Position Reconstruction
PRex	Position Reconstruction
SDS	Subsystem Design Specification
SFPRex	Single-Frame Position Reconstruction
SIS	Software Interface Specification
W1	WISE wavelength channel 1, 3.3 microns
W2	WISE wavelength channel 2, 4.7 microns
W3	WISE wavelength channel 3, 12 microns
W4	WISE wavelength channel 4, 24 microns
WISE	Wide-field Infrared Survey Explorer
WSDC	WISE Science Data Center
WSDS	WISE Science Data System

2 Input

2.1 Control Input

SFPRex and MFPRex both read control input in the form of Fortran Namelist files and command-line parameters. For control parameters included in both command line and namelist inputs, the command line inputs override. Input FITS files and source detection files must be specified entirely via namelist or the command line, but not both, i.e., the two input methods may not be mixed for these files.

2.1.1 SFPRex Command-Line Parameters

The command-line parameters for SFPRex are given by its tutorial display:

```
sfprex: Single-Frame Position Reconstruction vsn 1.0  A71101

 usage:  sfprex <flags> <specifications>

 where <flags> <specifications> must be:

     -f1  FITSnam1   (FITS file name for W1)
     -f2  FITSnam2   (FITS file name for W2)
     -f3  FITSnam3   (FITS file name for W3)
     -f4  FITSnam4   (FITS file name for W4)
     -s1  SrcNam1    (Source file name for W1)
     -s2  SrcNam2    (Source file name for W2)
     -s3  SrcNam3    (Source file name for W3)
     -s4  SrcNam4    (Source file name for W4)
     -n   NLnam      (Namelist input file)
     -r   REFnam     (2MASS reference source file name; required)
     -o   OutFnam    (Output meta-data file name; required)
     -cm  chisqmax   (Chi-square maximum threshold for merging)
     -dr  dRA0       (Initial estimate of RA adjustment)
     -dd  dDec0      (Initial estimate of Dec adjustment)
     -dp  dPA0       (Initial estimate of PA adjustment)
     -v   RvBnam     (Reference vs. Band statistics file name)
     -p   PRPTSnam   (PRex merged source file name)
     -a              (Adjust frames to lowest band)
     -d   dist       (radial distance in arcsec for matching
                      sources for frame adjustment)
     -m   minMatch   (minimum number of matches to proceed
                      with frame adjustment)
     -z   zSigOff    (no. of sigma for rejecting matches used
                      for frame adjustment)
     -l              (Echo command line to stdout)
     -w              (Write namelist to stdout)

 Specifications must include a 2MASS reference input and at least
 one WISE band. FITS and Source inputs must be for matching WISE
 bands. Initial estimates of frame adjustments, dRA0, dDec0, and
 dPA0 are all in arcsec (dRA0 is in local true angular measure,
 i.e., not clock angle) and apply to the frame center.
```

2.1.2 SFPRex Namelist Parameters

The SFPRex module optionally reads a NAMELIST file. The name of this file must be given on the command line via the "-n" option. The name of the NAMELIST is sfprin. The parameters defined in the NAMELIST are as follows.

Name	Description	Dim	Type	Units	Default
chsqmx	Maximum value allowed for the chi-square value based on source position discrepancy for merging	1	R*8	-	6.0d0
drwin	Search window for matches in the pattern-matching processing	1	R*8	asec	8.0d0
dsf1st	Flags to control whether to compute scale factor for corresponding band before general least-squares fitting	4	L*4	-	2*T,2*F
dstron	Flag to specify whether to include distortion in coordinate transformations	1	L*4	-	T
dsyeqx	Flag to specify whether to force X and Y scale factors to be equal in given band	4	L*4	-	4*F
FITSnam	Names of input FITS files, one per band	4	C*200	-	<none>
fixed	Array of flags controlling which of the 20 fit para-are held constant	20	L*4	-	13*F,2*T, 3*F,2*T
fracmn	Minimum fraction of sources that are matched to sources in other bands	1	R*8	-	0.1d0
frmadj	If T, perform coarse frame-adjustment prior to merging WISE sources	1	L*4	-	F
ldepth	Look depth in pattern-match match search	1	I*4	-	99
MDnam	Name of meta-data output file	1	C*200	-	<none>

Name	Description	Dim	Type	Units	Default
MinMatch	Minimum number of matches between two bands in frame-adjustment processing	1	I*4	-	3
mwtfp0	Multipliers on weight of a priori band-frame position in least-squares fit, one per band	4	R*8	-	4*1.0d0
mwf2f0	Multipliers on weight of a priori band-frame position differences in least-squares fit, W1-W2, W1-W3, W1-W4, W2-W3, W2-W4, W3-W4	6	R*8	-	6*1.0d0
mxpran	Maximum allowed probability that all pattern-matches are random	1	R*8	-	1.0d-5
NamWrt	If T, the sfprin namelist will be written to stdout after it is read and all command-line specifications have been processed	1	L*4	-	F
nmin	Minimum number of points to to compute a trimmed mean	1	I*4	-	3
PRPTSnam	Merged-source output file name	1	C*200	-	<none>
PSnam	Names of input source files, one per band	4	C*200	-	<none>
radist	Radial distance for search for matching sources in different bands in frame adjustment processing	1	R*8	asec	5.0d0
REFnam	Name of input 2MASS reference sources	1	C*200	-	<none>
RvBnam	Reference vs. Band statistics output file name	1	C*200	-	<none>
sepmin	Minimum separation between peg stars in pattern match processing	1	R*8	asec	60.0d0
tblX	Table-file column name for source X position	1	C*25	-	‘x’

Name	Description	Dim	Type	Units	Default
tblY	Table-file column name for source Y position	1	C*25	-	‘y’
tblSigX	Table-file column name for source X position uncertainty, 1-sigma	1	C*25	-	‘sigx’
tblSigY	Table-file column name for source Y position uncertainty, 1-sigma	1	C*25	-	‘sigy’
tblSigXY	Table-file column name for source XY position uncertainty co-sigma	1	C*25	-	‘sigxy’
tblF	Table-file column name for source flux parameter	1	C*25	mag	‘mag’
toldpa	Maximum PA difference for separation bar match in the pattern matching	1	R*8	asec	4.0d3
tolds	Maximum scale factor difference for separation bar match in the pattern matching	1	R*8	asec	1.5d-2
twkmch	If T, tweak source positions with current solution and recompute matches in the pattern match processing	1	L*4	-	T
useals	If T, use all two-peg pattern match results with the same best match count	1	L*4	-	T
usebnd	If T, include corresponding band in processing, one per band	1	L*4	-	4*T
usef2f	If T, use corresponding band-frame position differences in least-squares fit, W1-W2, W1-W3, W1-W4, W2-W3, W2-W4, W3-W4	6	L*4	-	6*T
usef2r	If T, use corresponding band-2MASS position differences in least-squares fit, one per band	4	L*4	-	T,T,F,F

Name	Description	Dim	Type	Units	Default
ZsigOff	Maximum allowed fluctuation in sigma units for: (a.) dispersion in frame adjustment measurements relative to mean source RSS uncertainty; (b.) discrepancy between seed and adjusted candidate position per axis for inclusion in frame adjustment averaging	1	R*8	-	5.0d0

2.1.3 MFPRex Command-Line Parameters

TBS

2.1.4 MFPRex Namelist Parameters

TBS

2.2 FITS Input

SFPRex and MFPRex both read FITS files corresponding to WISE level-1 images. Information on the celestial coordinates, scale factors, and distortion is obtained from the headers. Different combinations of input parameters may be used. The keywords checked are: CRVAL1, CRVAL2, CRPIX1, CRPIX2, CD1_1, CD1_2, CD2_1, CD2_2, CROTA2, CRDER1, CRDER2, UNCRTPA, CDELT1, CDELT2, CSDRADEC, A_0_0, A_0_1, A_0_2, A_0_3, A_1_0, A_1_1, A_1_2, A_1_3, A_2_0, A_2_1, A_2_2, A_2_3, A_3_0, A_3_1, A_3_2, A_3_3, B_0_0, B_0_1, B_0_2, B_0_3, B_1_0, B_1_1, B_1_2, B_1_3, B_2_0, B_2_1, B_2_2, B_2_3, B_3_0, B_3_1, B_3_2, B_3_3, AP_0_0, AP_0_1, AP_0_2, AP_0_3, AP_1_0, AP_1_1, AP_1_2, AP_1_3, AP_2_0, AP_2_1, AP_2_2, AP_2_3, AP_3_0, AP_3_1, AP_3_2, AP_3_3, BP_0_0, BP_0_1, BP_0_2, BP_0_3, BP_1_0, BP_1_1, BP_1_2, BP_1_3, BP_2_0, BP_2_1, BP_2_2, BP_2_3, BP_3_0, BP_3_1, BP_3_2, and BP_3_3.

2.3 Source Detections Input

SFPRex and MFPRex both read source detection table files conforming to the WISE SIS TBS. These extractions are generated by an upstream module (mdet) in the single-scan pipeline. Each file must specify the number of sources in the header on a line beginning "\Nsrcs =". For each detection, the X and Y pixel positions and uncertainties are read, along with the flux magnitude. Technically the source extraction program produces (RA, Dec) positions which are then converted back to (X, Y) frame positions for each band in the Perl wrapper. The default column names are x, y, sigx, sigy, sigxy, mag. These may be specified via namelist (see section 2.1.2

above, parameters tblX, tbkY, tblSigX, tblSigY, tblSigXY, and tblF. All parameters are required except for the co-sigma uncertainty sigxy, which will be set to zero if absent and otherwise will be used as part of the error covariance matrix in all Gaussian chi-square tests and refinement operations (note: the co-sigma is related to the covariance as follows: the covariance is equal to the co-sigma multiplied by the absolute value of the co-sigma).

2.4 2MASS Astrometric Reference Input

SFPRex and MFPRex both read 2MASS astrometric reference data conforming to the SIS TBS. These 2MASS source records are provided by a database-access utility upstream in the single-scan pipeline and formatted into a table file. The header must have a line giving the number of sources beginning \ RowsRetreived =" [sic], and the columns must contain the parameters ra, dec, err_maj, err_min, err_ang, and k_m.

A subset of the 2MASS Point Source Catalog has been selected for use as the astrometric reference for WISE position reconstruction. That is what is being referred to when the term "2MASS" is used. The current selection criteria are very conservative, taking clean sources with Ks magnitudes between 5.5 and 12.0. This results in 30 million sources, or an average count of 446 per WISE frame. Of course, the actual counts vary with sky position, and not all are visible at WISE wavelengths. Work is needed to determine what fraction can be expected to be visible at each of the WISE wavelengths. If it should turn out that more 2MASS sources are needed, the Ks magnitude range could be expanded to go from 4.5 to 14.5 while still keeping the accuracy below 0.1 arc-second.

A major issue which remains to be resolved is the impact of proper motions on the reconstruction accuracy of the band frames. It's been a decade now since the first 2MASS observations were made, and the 2MASS catalog does not include proper motions. If proper motions were random, this would not be a problem, because of the large average number of sources involved in the reconstruction of each frameset. Unfortunately, proper motions are correlated due to solar motion through space, and the extent of that correlation depends on sky position relative to the vector which defines that motion. Just how significant are these correlations, over what portion of the sky? The options for dealing with this range from trying to determine proper motions for the selected 2MASS stars (a very large task, probably beyond the present scope) to trying to characterize proper motions statistically as a function of sky position and then adding in appropriate uncertainties.

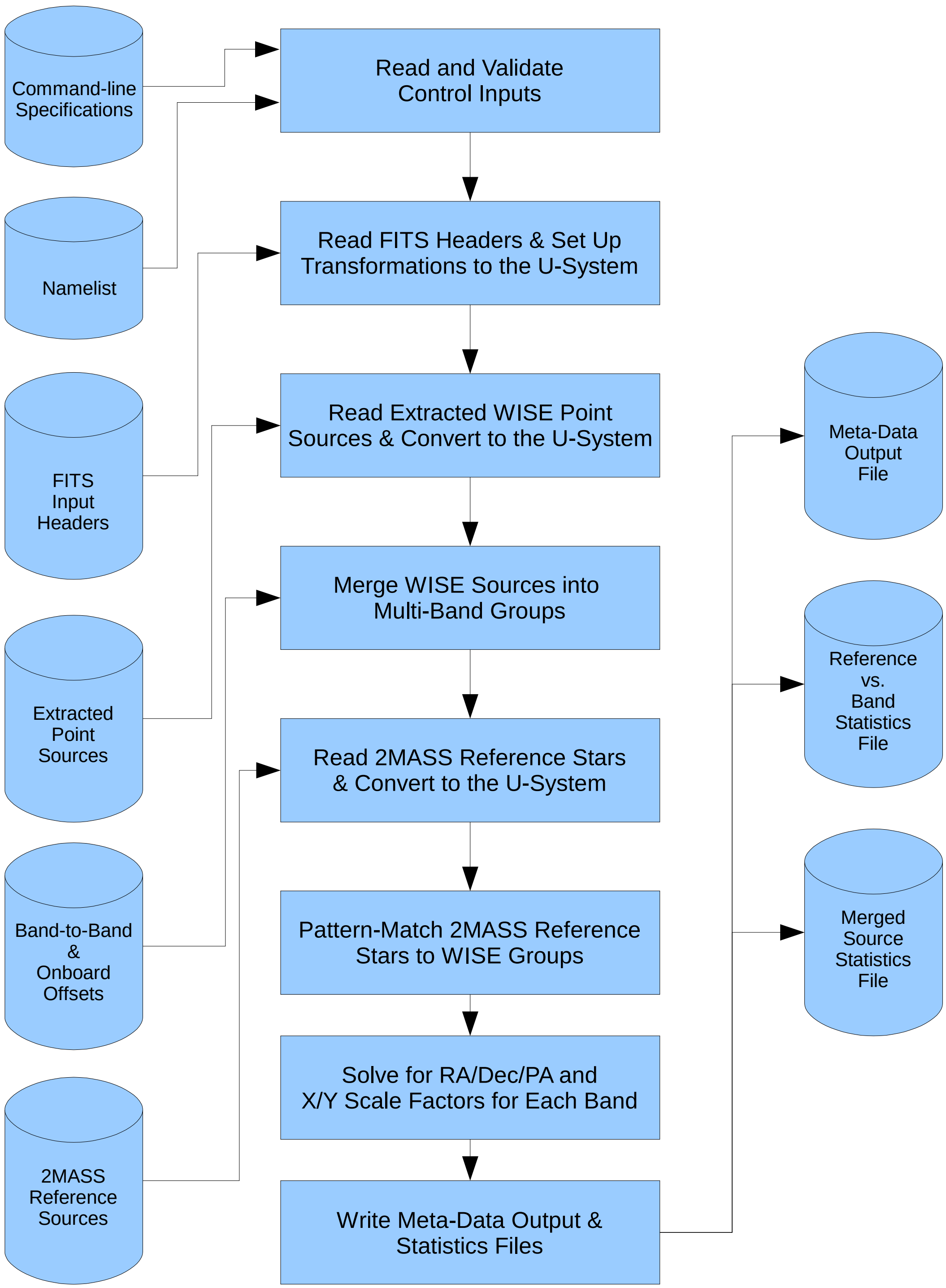

Figure 1. Overall SFPRex Functional Flow Diagram

3 Processing

3.1 SFPRex Processing

The high-level functional flow of the SFPRex module is shown in Figure 1.

3.1.1 Initialization

The SFPRex module initializes itself by:

A.) reading and processing its control inputs;

B.) reading the FITS headers for the four (or fewer, TBD) frames being processed;

C.) reading in all WISE source extraction records for each frame;

D.) reading in the 2MASS astrometric reference records;

E.) setting up the U (Universal) coordinate system based on the adjusted *a priori* band-frame center of shortest wavelength band used.

.

The WISE and 2MASS sources are transformed (including distortion correction for the WISE sources) into the U-system. The U-system is centered at the adjusted *a priori* center position of the shortest wavelength band in use. Its y-axis points north, it's x-axis west and its units are always arcseconds. Once the U-system is set, it remains locked in that sky position regardless of any band-frame position adjustments made during the SFPRex processing.

The native coordinate system for a group of source detections in a given band is called the *band-frame* coordinate system. This is in units of pixels and maps directly to the individual FITS images in each band. The band-frame coordinate systems for different bands do not share a common zero point, plate scale, or twist angle. In other words, each is related to the U-system by five parameters:

X_{uo}	X offset in arcsec of band-frame origin from U-system origin
Y_{u0}	Y offset in arcsec of band-frame origin from U-system origin
θ	Rotation in degrees of band-frame axes relative to U-system axes
s_x	X scale factor to convert from band-frame to U-system
s_y	Y scale factor to convert from band-frame to U-system

To distinguish between WISE bands, a superscript prefix *i* is used; e.g., ${}^{i}X_{u0}$ is the X offset of the band-frame origin for band *i* relative to the U-system origin, etc.; nominally *i* runs from 1 to 4. By default, distortion is corrected directly in band-frame coordinates. After that, U-system coordinates (X_u, Y_u) are computed from band-frame coordinates (X_f, Y_f) in band *i* as follows.

$$\begin{pmatrix} {}^iX_u \\ {}^iY_u \end{pmatrix} = \begin{pmatrix} {}^iX_{u0} \\ {}^iY_{u0} \end{pmatrix} + \begin{pmatrix} \cos({}^i\theta) & \sin({}^i\theta) \\ -\sin({}^i\theta) & \cos({}^i\theta) \end{pmatrix} \begin{pmatrix} {}^is_x \; {}^iX_f \\ {}^is_y \; {}^iY_f \end{pmatrix} \tag{1}$$

The rotation matrix in this equation will be denoted ${}^iT_f^u$ below, or just T_f^u when band distinction is not needed. The goal of the subsequent processing is to match sightings of the various stars, in both 2MASS and the various WISE bands, to identify which observations apply to the same object and deduce corrections to the five parameters above for each band processed from the position discrepancies. The purpose of the U system is to provide a common coordinate system within which meaningful analysis of the positions of sources in different bands may be performed. The general case involves four WISE bands, and so a 20x20 system of equations is to be solved (see Appendix A for reduced-size systems). First the WISE detections are merged into "groups", because the large possible telescope pointing errors should not disrupt the positions of the WISE detections relative to each other. After the WISE sources are grouped, pattern matching between them and the 2MASS sources is performed, followed by band-frame position refinement.

In order to be done optimally, the matching must take into account the uncertainties in the source positions (X_f, Y_f), where the band index i is suppressed here and for the remainder of this section to reduce clutter. The 1-σ uncertainties of these coordinates in the band-frame system are σ_{xf} and σ_{yf}, respectively; the covariance between errors in these coordinates is expressed as a *co-sigma*, σ_{xyf}, related to the covariance V_{xyf} by the formula $V_{xyf} = \sigma_{xyf} |\sigma_{xyf}|$. This maintains the sign information for the correlation, since σ_{xyf} may be negative. It is more natural to carry the co-sigma along with the other uncertainties instead of the covariance because the former is in the same units as the other uncertainties. The diagonal elements of the position error covariance matrix are $V_{xf} = \sigma_{xf}^2$ and $V_{yf} = \sigma_{yf}^2$. The error covariance matrix is converted from band-frame coordinates to the U system via a similarity transformation as follows.

$$\Omega \equiv \begin{pmatrix} V_{xu} & V_{xyu} \\ V_{xyu} & V_{yu} \end{pmatrix} = \left(T_f^u\right)^T \begin{pmatrix} s_x^2 V_{xf} & |s_x s_y| V_{xyf} \\ |s_x s_y| V_{xyf} & s_y^2 V_{yf} \end{pmatrix} T_f^u \tag{2}$$

This results in the following uncertainties in the U system.

$$\begin{aligned}
\sigma_{xu} &= \sqrt{V_{xu}} = \sqrt{s_x^2 V_{xf} \cos^2\theta + s_y^2 V_{yf} \sin^2\theta - 2|s_x s_y| V_{xyf} \sin\theta\cos\theta} \\
\sigma_{yu} &= \sqrt{V_{yu}} = \sqrt{s_y^2 V_{yf} \cos^2\theta + s_x^2 V_{xf} \sin^2\theta + 2|s_x s_y| V_{xyf} \sin\theta\cos\theta} \\
V_{xyu} &= (s_x^2 V_{xf} - s_y^2 V_{yf}) \sin\theta\cos\theta + |s_x s_y| V_{xyf} (\cos^2\theta - \sin^2\theta) \\
\sigma_{xyu} &= sign(V_{xyu}) \sqrt{|V_{xyu}|}
\end{aligned} \tag{3}$$

3.1.2 Group Merging

The WISE sources are arranged into "merge groups" as follows. If only one WISE band is present, each source is taken as a "group" and used as is, i.e., the group positions, uncertainties and fluxes are those of the WISE sources. If two or more bands are present, the more general processing described below is performed.

If "frame adjustment" was specified ("-a" on the command line, or "`frmadj = T`" in namelist), then frame adjustment preprocessing is performed. This involves position-registering bands of wavelength higher than the lowest-wavelength band present (nominally W1) to the lowest, i.e., removing mean position offsets (this is expected to be needed only during in-orbit checkout). The lowest-wavelength WISE band available is taken as the "seed" band, and each higher-wavelength band is processed as a "candidate" band. For each detection in the seed band, all detections in the the other candidate bands are examined; if one and only one candidate detection per band is within the match radius "`radist`" (namelist, default 5 arcsec, specifiable on the command line via "-d"), then the match is accepted; if all candidate-band matches to the seed are unconfused (i.e., no band has more than one detection in the seed's match area), then the position offsets are used in summations employed to compute mean candidate-to-seed offsets and RMS dispersions.

After all matching has been done for a given seed-candidate band combination, the number of matches is counted and must be at least equal to the namelist parameter `MinMatch` (default 3, command-line specification "-m"); if this is not satisfied, then `radist` is enlarged by a factor of 1.1 and the seed-candidate bands are reprocessed. This may be done up to 100 times, after which if it still has not succeeded, the merge subroutine quits and returns an error indication. If it succeeds, then the dispersions about the mean offsets are compared to the RMS seed+candidate variances for matched sources, and the former must be no more than `ZsigOff` times the latter on each axis (see section 2.1.2; command-line specification "-z"; default 5 arcsec). If this test fails, the seed-candidate processing is performed again from the beginning, but this time individual matches are rejected if the offset on either axis is larger than `ZsigOff` times the RMS seed+candidate variance from the first pass. If all tests are passed in all seed-candidate combinations, then the mean band-to-band offsets are subtracted off of the position differences used in the subsequent chi-square tests (note that the candidate-band positions themselves are not altered, since they are needed for subsequent analysis).

After frame adjustment, if any, the chi-square match testing is performed for all band combinations. Position errors are assumed to be Gaussian, and so Gaussian statistics are employed for matching and position refinement of matched sources forming a group. Groups are sought which contain one and only one source in each band, each of which passes a standard chi-square position discrepancy test with at least one other source in the group (not necessarily all), and if any source has more than one acceptable match in any other band, the entire group is rejected, and its sources may not be used in any other groups. Using the notation of Equations (1), (2), and (3) (see section 3.1.1), but suppressing the *u* subscript with the understanding that all computations here are in the U system, and attaching indexes *i* and *j* to indicate bands, and *m* and *n* to indicate detection numbers within bands, the chi-square for a pair of detections comprised of detection *m* in band *i* and detection *n* in band *j* is computed as follows; the frame-adjustment offsets in X and Y are denoted δx_i and δy_i, respectively; these are zero for $i = 1$, and all other offsets are also zero if frame adjustment has *not* been performed.

$$
\begin{aligned}
&\Delta X = ({}^{i}X_m - \delta x_i) - ({}^{j}X_n - \delta x_j) \\
&\Delta Y = ({}^{i}Y_m - \delta y_i) - ({}^{j}Y_n - \delta y_j) \\
&\vec{\Delta} \equiv (\Delta X, \Delta Y) \\
&\Omega = {}^{i}\Omega_m + {}^{j}\Omega_n = \begin{pmatrix} {}^{i}V_{xm} & {}^{i}V_{xym} \\ {}^{i}V_{xym} & {}^{i}V_{ym} \end{pmatrix} + \begin{pmatrix} {}^{j}V_{xn} & {}^{j}V_{xyn} \\ {}^{j}V_{xyn} & {}^{j}V_{yn} \end{pmatrix} \equiv \begin{pmatrix} V_x & V_{xy} \\ V_{xy} & V_y \end{pmatrix} \\
&W \equiv \Omega^{-1} = \frac{1}{D}\begin{pmatrix} V_y & -V_{xy} \\ -V_{xy} & V_x \end{pmatrix} \equiv \begin{pmatrix} W_x & W_{xy} \\ W_{xy} & W_y \end{pmatrix}, \quad D \equiv V_x V_y - V_{xy}^2 \\
&\chi^2 = \vec{\Delta} W \vec{\Delta}^T = W_x \Delta X^2 + W_y \Delta Y^2 + 2 W_{xy} \Delta X \Delta Y \\
&\qquad = \frac{V_y \Delta X^2 + V_x \Delta Y^2 - 2 V_{xy} \Delta X \Delta Y}{D}
\end{aligned}
\tag{4}
$$

Any given pair of sources in different bands is considered to have passed the chi-square test if the value of their 2-D chi-square is found to be less than or equal to the threshold `chsqmx` (default 6.0, which rejects 5% of all true matches and is chosen for reliability more than completeness; see section 2.1.2). The numbers of confused sources and orphan sources are counted and reported in the meta-data output file (see section 4.1.1).

After the groups have been formed, if any, the refined position for each group is computed by normal Gaussian refinement as follows, where $\{G\}$ indicates the set of band numbers represented in the group; for example, {1,2,4} corresponds to a group with detections in W1, W2, and W4. Since each band present is represented by only one detection, we suppress the detection-within-band index; e.g., ${}^{i}V_x$ is the *X* uncertainty variance in U-system coordinates for the group's detection in band *i*, whatever its index within that band may be.

$$
\begin{aligned}
&W = \sum_{i \in \{G\}} {}^{i}\Omega^{-1} = \sum_{i \in \{G\}} \frac{1}{{}^{i}V_x\,{}^{i}V_y - {}^{i}V_{xy}^2} \begin{pmatrix} {}^{i}V_y & -{}^{i}V_{xy} \\ -{}^{i}V_{xy} & {}^{i}V_x \end{pmatrix} \\
&\Omega_{refined} = W^{-1} \\
&\begin{pmatrix} X \\ Y \end{pmatrix}_{refined} = \Omega_{refined} \sum_{i \in \{G\}} {}^{i}\Omega^{-1} \begin{pmatrix} {}^{i}X \\ {}^{i}Y \end{pmatrix}
\end{aligned}
\tag{5}
$$

Note that if frame adjustment has been specified, the adjusted band positions are used to compute the weighted average merge group positions. This means that the merge group positions are tied to the shortest wavelength band, but have the advantage of including sources which appear only in longer wavelength bands. The flux reported for the group is the brightest of any of the sources in the group (i.e., minimum magnitude). The average offsets of the group from the seed band (lowest wavelength) are computed for both axes.

3.1.3 Pattern Matching

The pattern matching algorithm uses "separation bar" matching to identify potential star pairs common to 2MASS and the WISE groups. Figures 2 and 3 illustrate a simple case involving four 2MASS stars and six WISE merged groups, respectively. In practice, not all 2MASS stars and WISE groups are used; the depth is determined by the `ldepth` parameter (see section 2.1.2), which defaults to 99, meaning that only the brightest 99 2MASS and WISE groups are used.

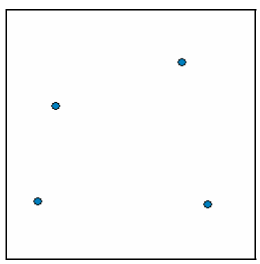

Figure 2 2MASS Reference Stars

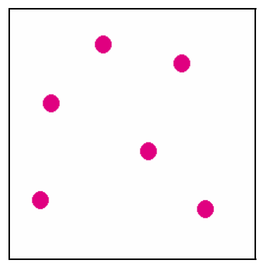

Figure 3 WISE Merge Groups

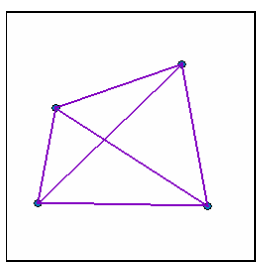

Figure 4 2MASS Reference Star Bars

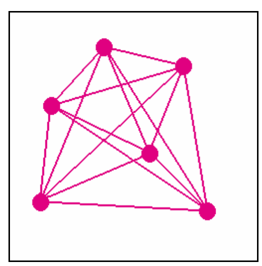

Figure 5 WISE Merge Group Bars

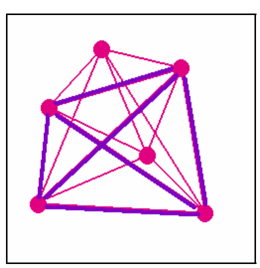

Figure 6 2MASS Bars Identified in Group Bars

Each 2MASS star used for mattern matching is connected to each other one by a "separation bar", as shown in Figure 4; the same is done for the WISE merged groups, as shown in Figure 5. Each bar is characterized by a slope and a length. The location of the endpoints is not considered significant, since the point of this analysis is to discover rather large offsets between the 2MASS and WISE patterns. The goal is to identify which separation bars are the same for the two groups of stars, as illustrated in Figure 6. For this simplified example all the 2MASS separation bars also show up as WISE separation bars. In general this will not be the case.

For N stars (of either type), there are $N(N-1)$ separation bars (i.e., the same as the number of elements above the diagonal of a matrix). All such bars are processed by computing the slope and length and recording the ID numbers of the sources at both ends of the bar. In order to be usable, a bar must have a length of at least `sepmin` (see section 2.1.2; default is 60 arcsec). The parameters for sufficiently long bars are stored in arrays to produce two sets of data, one for the 2MASS reference stars and one for the WISE groups. The stars defining the endpoints of the bar are stored with the one closest to the center of the array first; this is done so that endpoint star pairings between 2MASS and WISE may be assumed later to be such that the first entries correspond to each other and the second entries correspond to each other (the very small fraction of cases in which position errors invert this relationship are neglected and simply drop out of consideration).

Slope and length matches between these two data sets are then sought by considering each point in one data set in combination with each in the other as follows.

(a.) The ratio of the bar lengths is computed, subtracted from 1.0, and the absolute value is taken to obtain the *length ratio excess*.

(b.) The absolute difference in slope angle is computed in arcsec.

(c.) If the magnitude of the length ratio excess is not larger than the scale-factor-difference tolerance `tolds` and the absolute slope-angle difference is not larger than the angle-difference tolerance `toldpa` (see section 2.1.2; defaults are 0.015 and 4000 arcsec, resp.), then this pairing is stored in an array of potential separation bar matches.

d.) Each candidate separation bar match is then evaluated by forcing the end points of the WISE bar to match those of the corresponding 2MASS bar exactly; after the translation (ΔX and ΔY), rotation ($\Delta\theta$), and a single rescale factor *dsf* needed for this to happen are computed, they are applied to all the WISE sources; the total number (no `ldepth` limit) of source matches N_{pm} within the `drwin` tolerance is evaluated; this step is inherited from 2MASS where it was referred to as the "Two-Peg" pattern matcher.

e.) Point match counts and the associated WISE transformation parameters are stored in arrays for each candidate bar match pair; the best point match count N_{pmbest} is retained as the candidate bar matches are evaluated.

f.) The probability of achieving N_{pmbest} -2 point matches by random chance is evaluated via a Poisson model and compared to the maximum value allowed `mxpran`; if that value is exceeded, the routine is exited with "No Pattern Match Found"; otherwise processing continues.

g.) If the `useals` control parameter is set to T, then the four transformation parameters associated with all bar matches where the point match count is equal to N_{pmbest} are averaged, with outliers discarded; note that any successful pattern match is likely to have multiple separation bar matches associated with it; by averaging the results one can reduce the errors associated with a single set of end point matches.

h.) Further refinement is possible; if the `twkmch` control parameter is set to T, then all matched points are used to refine the transformation parameters further; a five-parameter least-square fit is performed, with the extra points allowing the X and Y scale factors to be corrected separately.

3.1.4 Parameter Fitting

Each source pair matched by the above processing is used to compute position discrepancies which contribute to summations of chi-squared values. These total chi-squares are then minimized by solving for corrections to the 20 parameters, i.e., the five per band listed in section 3.1.1. There are also *a priori* estimates of all 20 parameters and their uncertainties. Band-to-band and absolute relationships are included via proxy "measurements": three artificial sources "observed" in each band, that serve to constrain band-to-band changes, and terms that constrain absolute changes (i.e., changes involving the band-to-band relationships remaining constant but all parameters changing the same way in each band). While the total chi-square summation depends on the 20 parameters, it is convenient to split it up into four components: χ^2_{ww}, the part depending only on discrepancies between pairs of matched detections in different WISE bands; χ^2_{wr}, the part depending only on discrepancies between pairs consisting of a single-band WISE detection matched with a 2MASS reference source; χ^2_{aw}, the part depending only on discrepancies between pairs of "matched" proxy detections in different WISE bands; χ^2_{ar}, the part depending only on absolute changes to the 20 parameters. In each band, we write each of the five coordinate transformation parameters in Equation (1) as a sum of the current estimate for that parameter and a correction; the goal is to solve for these corrections by finding the values that minimize the total chi-square summation.

$$
\begin{aligned}
{}^{i}X_{u0} &\Leftarrow {}^{i}X_{u0} + {}^{i}\Delta X \\
{}^{i}Y_{u0} &\Leftarrow {}^{i}Y_{u0} + {}^{i}\Delta Y \\
{}^{i}\theta &\Leftarrow {}^{i}\theta + {}^{i}\Delta\theta \\
{}^{i}s_{x} &\Leftarrow {}^{i}s_{x}\left(1 + {}^{i}ds_{x}\right) \\
{}^{i}s_{y} &\Leftarrow {}^{i}s_{y}\left(1 + {}^{i}ds_{y}\right)
\end{aligned}
\tag{6}
$$

When these substitutions are made in Equation (1), the coordinates obtained in the U system become explicit functions of the five correction parameters. The first three are absolute correction factors indicated by the character Δ; the last two (for the scale factors) are relative correction factors indicated by the character *d*. The first chi-square summation is as follows.

$$
\chi^2_{ww} = \sum_{i=1}^{3}\sum_{j=i+1}^{4}\sum_{n=1}^{N_i}\sum_{m=1}^{Nj}\left[{}_{i}^{j}W_{xn}^{m}\left({}^{i}X_{un} - {}^{j}X_{um}\right)^2 + {}_{i}^{j}W_{yn}^{m}\left({}^{i}Y_{un} - {}^{j}Y_{um}\right)^2 + 2\,{}_{i}^{j}W_{xyn}^{m}\left({}^{i}X_{un} - {}^{j}X_{um}\right)\left({}^{i}Y_{un} - {}^{j}Y_{um}\right)\right]
\tag{7}
$$

This summation extends over all pairs of bands *i* and *j*, and within those bands, all pairs of detections *n* and *m*, respectively. The number of detections in bands *i* and *j* are N_i and N_j, respectively. The quantities summed are the individual chi-squares corresponding to the differences between each source pair's U-system X coordinates and Y coordinates, i.e., the products of all pair combinations of the coordinate differences weighted respectively as indicated by W_x, W_y, and W_{xy}, each with band prefixes *i* and *j* and source-number-within-band suffixes *n* and *m*. The weights are zero for all source pairs that are not matched; this implies also that the weights are zero for any pairs involving bands that are not being processed. For a pair of sources which *are* matched,

$$
\begin{aligned}
{}_{i}^{j}W_{xn}^{m} &= \frac{{}^{i}V_{yu_n} + {}^{j}V_{yu_m}}{\left({}^{i}V_{xu_n} + {}^{j}V_{xu_m}\right)\left({}^{i}V_{yu_n} + {}^{j}V_{yu_m}\right) - \left({}^{i}V_{xyu_n} + {}^{j}V_{xyu_m}\right)^2} \\
{}_{i}^{j}W_{yn}^{m} &= \frac{{}^{i}V_{xu_n} + {}^{j}V_{xu_m}}{\left({}^{i}V_{xu_n} + {}^{j}V_{xu_m}\right)\left({}^{i}V_{yu_n} + {}^{j}V_{yu_m}\right) - \left({}^{i}V_{xyu_n} + {}^{j}V_{xyu_m}\right)^2} \\
{}_{i}^{j}W_{xyn}^{m} &= \frac{-\left({}^{i}V_{xyu_n} + {}^{j}V_{xyu_m}\right)}{\left({}^{i}V_{xu_n} + {}^{j}V_{xu_m}\right)\left({}^{i}V_{yu_n} + {}^{j}V_{yu_m}\right) - \left({}^{i}V_{xyu_n} + {}^{j}V_{xyu_m}\right)^2}
\end{aligned}
\tag{8}
$$

where we have attached band prefixes and source-number-within-band suffixes to the uncertainty variances V_{xu}, V_{yu}, and V_{xyu} defined in Equations (2) and (3). These expressions result from expanding the chi-square for a full error-covariance matrix (i.e., nonzero off-diagonal element) in a manner analogous to Equation (4). Note that in the code, these summations are not actually carried out over all possible combinations as shown above; only the terms with nonzero weights are evaluated, and these are known from the group information which is stored in an array of detection pointers for each group. Similarly, an array of 2MASS reference associations with groups is maintained and used to evaluate the nonzero terms in the summation for the corresponding chi-square, where N_r is the number of reference stars.

$$\chi^2_{wr} = \sum_{i=1}^{4} \sum_{k=1}^{N_r} \sum_{n=1}^{N_i} \left[{}^iW^k_{xn} \left({}^iX_{un} - X_{uk} \right)^2 + {}^iW^k_{yn} \left({}^iY_{un} - Y_{uk} \right)^2 + 2\,{}^iW^k_{xyn} \left({}^iX_{un} - X_{uk} \right)\left({}^iY_{un} - Y_{uk} \right) \right] \tag{9}$$

The *a priori* constraints represent prior knowledge of the 20 parameters; they dilute the effects of any outlier measurements that may get through the filters, and they guarantee that the system of equations will be nonsingular by assuring that some measurement exists for every parameter. The *a priori* band-to-band constraints are implemented via three well spaced pseudo-sources defined in the band-frame coordinates of W1 at the following locations, where the index p denotes a "pseudo" or "prior" source.

$$\begin{aligned} {}^1_pX_{f1} &= 512.0\times {}^1s_x\ , \quad {}^1_pY_{f1} = 853.3\times {}^1s_y \\ {}^1_pX_{f2} &= 216.4\times {}^1s_x\ , \quad {}^1_pY_{f2} = 341.3\times {}^1s_y \\ {}^1_pX_{f3} &= 807.6\times {}^1s_x\ , \quad {}^1_pY_{f3} = 341.3\times {}^1s_y \end{aligned}$$

The *a priori* estimates of the 20 parameters can be used to map these three proxy sources from the W1 band-frame coordinates above to the U-system for the case being processed. From there, the coordinates in the other three WISE bands can be computed, again by using the *a priori* estimates of the 20 parameters. The result is three pseudo-groups, each with pseudo-detections in all bands. The third chi-square can then be written in a form analogous to Equation (7), noting that these proxy sources are not coupled in any way to real WISE detections or 2MASS references.

$$\chi^2_{aw} = \sum_{i=1}^{3} \sum_{j=i+1}^{4} {}^i_jK_{aw} \sum_{n=1}^{3} \sum_{m=1}^{3} \left[{}^j_iW^m_{pxn} \left({}^i_pX_{un} - {}^j_pX_{um} \right)^2 + {}^j_iW^m_{pyn} \left({}^i_pY_{un} - {}^j_pY_{um} \right)^2 + 2\,{}^j_iW^m_{pxyn} \left({}^i_pX_{un} - {}^j_pX_{um} \right)\left({}^i_pY_{un} - {}^j_pY_{um} \right) \right] \tag{10}$$

The factor ${}^i_jK_{aw}$ takes on the values specified by the namelist parameter `mwf2f0` (see section 2.1.2) for the given combination of i and j; it allows the weighting of these *a priori* constraints to be adjusted if necessary. For example, the number of proxy sources is taken to be three because that is the minimum needed to define all 20 parameters; but the number of WISE detections going into the fitting is not constant, and so the relative weighting of observed data and *a priori* data is not necessarily optimal with just three proxy sources. This adjustment factor allows a better estimate of the average weighting to be applied after some experience has been obtained with real data.

The fourth chi-square summation constrains absolute changes to the 20 parameters and is written explicitly in terms of them.

$$\chi^2_{ar} = \sum_{i=1}^{4} {}^iK_{ar} \left[\frac{{}^i\Delta X^2}{{}^iV_{x0}} + \frac{{}^i\Delta Y^2}{{}^iV_{y0}} + \frac{{}^i\Delta\theta^2}{{}^iV_{\theta 0}} + \frac{{}^ids_x{}^2}{{}^iV_{dsx0}} + \frac{{}^ids_y{}^2}{{}^iV_{dsy0}} \right] \tag{11}$$

The ${}^iK_{ar}$ factor corresponds to the namelist parameter `mwtfp0` (see section 2.1.2) and allows adjustment of the relative weighting of these constraints. The total chi-square is now:

$$\chi^2 = \chi^2_{ww} + \chi^2_{wr} + \chi^2_{aw} + \chi^2_{ar} \tag{12}$$

The substitutions in Equation (6) are made formally in Equations (1) and (7)-(11), and then the 20 derivatives are taken with respect to the 20 fitting parameters (the five corrections shown in Equation (6), one each for the five parameters listed in section 3.1.1, and for each of the four WISE bands, hence 20 correction parameters).

$$\begin{gathered} \frac{\partial \chi^2}{\partial P_{ik}} = 0\,,\; k = 1\; to\; 5,\; i = 1\; to\; 4 \\ P_{i1} \equiv {}^i\Delta X\,,\; P_{i2} \equiv {}^i\Delta Y\,,\; P_{i3} \equiv {}^i\Delta\theta,\; P_{i4} \equiv {}^ids_x\,,\; P_{i5} \equiv {}^ids_y \end{gathered} \tag{13}$$

With small-angle approximations for trigonometric functions of the angle corrections, and with second-order terms in the other corrections dropped, this produces a linear system of 20 equations in 20 unknowns which is solved with standard methods. For convenience, we remap the two indexes on the fitting parameters into one dimension, $n = 5(i\text{-}1)+k$. Then Equation (13) becomes

$$\frac{\partial \chi^2}{\partial P_n} = 0\,,\; n = 1\; to\; 20 \tag{14}$$

Expanding the squares in the definitions of the chi-square summations results in cross-terms in the five parameters per band, and then taking the derivatives as shown in Equation (14) produces the 20 equations in 20 unknowns which can be represented in vector-matrix form as $Ax = b$, where the components of the x vector are P_n, $n = 1, 20$, each row of the A matrix consists of the coefficients of P_n in the corresponding equation, and the b vector components are the constant terms in the 20 equations. The linearized system can be solved in a variety of ways, but the simplest method is to invert the A matrix, since that inverse is needed later anyway. Therefore we have $x = A^{-1}b$. The error covariance matrix expressing uncertainties in the 20 fit parameters is just the inverse of the coefficient matrix, $\Omega_P = A^{-1}$, where the elements of Ω_P are denoted V_{Pij}, the error covariance for parameters P_i and P_j. Thus $\sqrt{V_{P11}} = \sigma({}^1\Delta X)$, $V_{23} = \mathrm{cov}({}^1\Delta X, {}^1\Delta\theta)$, etc.

3.2 MFPRex Processing

TBS

4 Output

4.1 SFPRex Output

4.1.1 Meta-Data Output File

As originally envisioned, the meta-data output file was limited to the information used downstream to modify the headers of FITS files containing WISE array images. For each band processed, the values of refined RA, Dec, PA and scale factors for X and Y are written in table format. The meta-data scope is now extended to include a large number of SFPRex parameters (including both inputs and outputs) useful for historical tracking and statistical analysis. For details, see SIS TBD.

4.1.2 Reference vs. Band Statistics Output File

One file is generated for each of the four WISE bands when a file name is specified by the user via namelist (see `RvBnam` in section 2.1.2) or the command line (see "-v" in section 2.1.1); for a given band, the file name will have the band number concatenated to the end (e.g., if `RvBnam` is set to "rvbfile", then four files will be produced with the names "rvbfile1", "rvbfile2", "rvbfile3", and "rvbfile4"). Each file compares refined band source positions in U-system coordinates with associated 2MASS star positions, where available. The primary use for these files is to provide input to the offline distortion analysis program gnDSTR. For details, see SIS TBD.

4.1.3 PRex Merged Source Output File

This is a file of refined band-merged (within SFPRex) WISE source positions in RA and Dec coordinates. Its primary use is for QA. It can be used to evaluate frameset overlap differences as well as differences with respect to external catalogs. It is not used during the position reconstruction. For details, see SIS TBD.

4.2 MFPRex Output

TBS

5 Testing and Parameter Tuning

5.1 SFPRex Testing

The sfprex module has been unit-tested with simulated data hand-edited to force the various processing paths to be traversed. It has also been tested with simulated FITS data generated by N. Wright for the galactic center, with sources extracted by the WSDS sdex module and real 2MASS reference stars retrieved from the 2MASS Point Source Catalog. Additional unit testing will be needed after the 20-parameter frameset fit routine is in place and a better understanding has been obtained regarding the number of available sources per band and those that can be associated between bands. In particular, the effect of low W3 and W4 source counts and low matched source counts between the longer-wavelength bands (W3 and W4) and the shorter-wavelength bands (W1 and W2) needs to be explored. Also, the effect of very low match counts between W3/W4 sources and 2MASS should be considered. It is expected that if W1 and W2 detectors are lost, the WISE accuracy requirements will not be achievable.

5.2 SFPRex Parameter Tuning

There are several SFPRex input parameters which will require tuning prior to their use in routine pipeline processing. These parameters fall into two categories.

The first category is composed of parameters used to control the pattern matcher. These parameters have to be tuned such that it is unlikely that a good match will be rejected and very unlikely that a bad match will be accepted. The latter case is particularly to be avoided because of the downstream repercussions. Setting the parameters too far the other way will also cause problems by requiring manual intervention. After initial tuning, the proper balance was achieved in the 2MASS processing with a similar pattern matcher.

The second category is composed of the ${}_{j}^{i}K_{aw}$ weighting factors used with the proxy sources in Equation 10. These need to be set such that under ordinary circumstances they have a negligible effect on the solution. But in a case where a band (such as W4) has no source matching to other bands or 2MASS, its solution can be pulled along with those of the other bands. A good portion of the parameter tuning can be done using simulation data, but the final tuning, particularly for the pattern-match control parameters, will have to await IOC data.

5.3 MFPRex Testing

TBS

5.4 MFPRex Parameter Tuning

TBS

Appendix A. Constrained Solutions and Reduced Systems of Equations

It is sometimes useful to constrain one or more parameters in the chi-square minimization fit, resulting in a reduced set of simultaneous equations. Within PRex, two different types of constraint are available. First, a subset of the nominal fit parameters can be selected as `fixed` (see section 2.1.2, SFPRex Namelist Parameters), meaning that they will be frozen at their *a priori* values. Second, the fractional scale changes (ds_x and ds_y) within any band frame can specified to be the same. This option is controlled using the namelist array `dsyeqx`.

A.1 Fixed Parameters

In order to fix user-specified parameters, the rows in the matrix equation $Ax = b$ associated with those fixed parameters are dropped. The columns in the A matrix corresponding to the fixed parameters are multipled by the fixed parameter values, then moved accross the equal sign to be subtracted from the b vector. As a simplified example, consider a single-band case resulting in the following matrix equation:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{pmatrix} \Delta X \\ \Delta Y \\ \Delta\theta \\ ds_x \\ ds_y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} \tag{A.1}$$

Assume it is desired to fix ds_x and ds_y at their *a priori* values ds_{x0} and ds_{y0}; the reduced matrix equation becomes:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \Delta X \\ \Delta Y \\ \Delta\theta \end{pmatrix} = \begin{pmatrix} b_1 - a_{14}\, ds_{x0} - a_{15}\, ds_{y0} \\ b_2 - a_{24}\, ds_{x0} - a_{25}\, ds_{y0} \\ b_3 - a_{34}\, ds_{x0} - a_{35}\, ds_{y0} \end{pmatrix} \tag{A.2}$$

A.2 Scale Changes Equal

Forcing the x and y scale changes to be equal (solving for $ds \equiv ds_x = ds_y$) can be achieved by adding the rows and the columns associated with scale changes. This will reduce the set of simultaneous equations by one for each band where this option is selected. As an example, consider once again the single-band case, which now becomes:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14}+a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24}+a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34}+a_{35} \\ a_{41}+a_{51} & a_{42}+a_{52} & a_{43}+a_{53} & a_{44}+a_{45}+a_{54}+a_{55} \end{pmatrix} \begin{pmatrix} \Delta X \\ \Delta Y \\ \Delta \theta \\ ds \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4+b_5 \end{pmatrix} \tag{A.3}$$